

# Strategic Learning and the Topology of Social Networks

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## Abstract

We consider a Bayesian game of pure informational externalities, in which a group of agents learn a binary state of the world from conditionally independent private signals, by repeatedly observing the actions of their neighbors in a social network.

We show that the question of whether or not the agents learn the state of the world depends on the topology of the social network. In particular, we identify a geometric “egalitarianism” condition on the social network graph that guarantees learning in infinite networks, or learning with high probability in large finite networks, in any equilibrium of the game. We give examples of non-egalitarian networks with equilibria in which learning fails.

**Keywords:** Bayesian learning, rational expectations, informational externalities, social networks, aggregation of information.

## 1 Introduction

We consider a Bayesian game played on a social network. There is an unknown state of the world  $S \in \{0, 1\}$ , and each of a finite or countably infinite group of agents receives a conditionally i.i.d. signal that carries information on  $S$ . In each discrete time period, each agent chooses an action in  $\{0, 1\}$ , where the utility is one if the action equals  $S$  and zero otherwise, and is discounted exponentially, by a common rate. The agents gain no further direct indication of the merit of their actions, but may learn by observing the actions of their neighbors in a social network. Externalities in this game are therefore purely informational; an agent’s utility does not depend directly on the actions of others, and so agents play sub-optimally only in hope of extracting additional information from their neighbors’ future actions.

This game can be thought of as modeling repeated lifestyle choices, where much can be learned from others, and utilities (e.g., longevity) are only revealed after a long amount of time. Alternatively, consider the choice of religion: this is evidently a repeated choice which is susceptible to social influence, and, arguably, the payoff is only delivered in the afterlife<sup>1</sup>.

We consider the question of *learning*. When the number of agents is infinite, when is it the case that the agents learn  $S$  and all converge to the correct action? In finite networks, when does learning happen with high probability? We show that learning may or may not occur, depending on the geometry of the social network. In particular, we show that in infinite networks that are egalitarian - in a sense we define below - learning occurs with probability one in any equilibrium. We require the

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distributions of private signals to be non-atomic, but do not impose unbounded likelihood ratios: learning occurs in egalitarian networks even for (informative) signals with bounded likelihood ratios. We likewise show that, for large, egalitarian, finite networks, learning occurs with high probability in any equilibrium; under a uniform egalitarianism constraint, as the size of the network tends to infinity, the probability of learning approaches one. We provide examples of non-learning in non-egalitarian networks.

We represent social networks as directed graphs. The nodes of the graph are the agents, and there is an edge between agent  $i$  and  $j$  if  $i$  observes the actions of  $j$ . The social networks we consider are *strongly connected*: for each pair of agents  $i$  and  $j$  there is a directed path from  $i$  to  $j$  in the social network graph. The *out-degree* of agent  $i$  is the number of agents it observes. We assume that it is finite, but allow infinite *in-degrees*, so that some agents may be observed by infinitely many others.

We show that learning occurs in infinite graphs that are egalitarian in the following sense. First, we require that out-degrees are bounded: there is some number  $d$  such that no agent observes more than  $d$  others. Second, we require that there exists a number  $L$  such that, if there is a path of length  $k$  from agent  $i$  to agent  $j$ , then there is a path from  $j$  back to  $i$ , of length at most  $L \cdot k$ . In this case, we call the graph  $L$ -connected. Equivalently, a graph is  $L$ -connected if, whenever  $i$  observes  $j$ , then there is a path of length at most  $L$  from  $j$  back to  $i$ . In a sense, and similarly to the bounded degree requirement,  $L$ -connectedness ensures that the “importance” of different agents does not vary too much; an agent is never too far removed from those who observe it. We therefore think of both of these conditions as describing a network that is, to a certain extent, egalitarian. From a mathematical viewpoint, these conditions arise naturally as a compactness condition in a certain topology of graphs.

We provide two example of non-learning. In the first example the social network graph has bounded out-degrees, but is not  $L$ -connected. We show here that, for a low enough discount factor, myopic behavior is an equilibrium that leads to convergence to the wrong action with probability that is small, but bounded away from zero.

In the second example the graph is undirected, so that whenever  $i$  observes  $j$  then  $j$  observes  $i$ ; the graphs is therefore 1-connected. However, it does not have bounded degrees, and so is not “egalitarian”. Here we construct a rather involved equilibrium, with non-myopic behavior, which again leads to non-learning with probability bounded away from zero.

Learning on social networks is a widely studied field; a complete overview is beyond the scope of this paper, and so we shall note only a few related studies. Bala and Goyal [4] study a similar model, and show results of learning or non-learning in different cases. Their model is crucially different from ours, making the results incomparable: it is boundedly-rational, and so agents do not take into account the choices of their neighbors when forming their beliefs. In their model, agents gain additional information directly by taking certain actions, and so their strategic behavior is a product of an individual explore-exploit dilemma, rather than social manipulation.

Other notable models of learning through repeated social interaction are those of DeGroot [9], Ellison and Fudenberg [11], DeMarzo, Vayanos and Zwiebel [10] and Golub and Jackson [14]. All of these papers describe models of agents that are not rational but rather use some rule of thumb. In fact, to the best of our knowledge, no previous work considers learning, in repeated interaction, on social networks, in a fully rational, strategic setting. In a previous paper [19], we consider the same question, but for myopic agents, who, again, display no strategic behavior.

The study of *agreement* (rather than learning) on social networks is also related to our work, and in fact we make crucial use of the work of Rosenberg, Solan and Vieille [24], who prove an agreement result for a large class of games with informational externalities played on social networks. This is a field of study founded by Aumann’s “Agreeing to disagree” paper [3] and elaborated on

by Sebenius and Geanakoplos [25], McKelvey and Page [16], Parikh and Krasucki [23], Gale and Kariv [12], Ménager [17] and Mueller-Frank [22], to name a few. The moral of this research is that, by-and-large, rational agents eventually reach consensus, even in strategic settings. Indeed, the fact that agents eventually agree is a condition for learning, since otherwise it is impossible that all converge to the same action. We elaborate on the work of [24] and show that when private signals are non-atomic then, asymptotically, agents agree on best responses.

Another strain of related literature is that of *herd behavior*, started by Banerjee [5] and Bikhchandani, Hirshleifer and Welch [7], with significant generalizations and further analysis by Smith and Sørensen [26], Acemoglu, Dahleh, Lobel and Ozdaglar [1] and recently Lobel and Sadler [15]. Here, the state of the world and private signals are as in our model, and agents are rational. However, in these models agents act sequentially rather than repeatedly, and so no non-trivial strategic behavior arises.

In Section 2 we define our model formally and state our results. In Section 3 we prove our results, and in Section 4 we provide examples of non-learning.

## 2 Model and results

### 2.1 Definitions

The following definition is adapted from [21].

**Definition 2.1.** *Let  $\Omega$  be a measurable space of **private signals**, and let  $\mu_0$  and  $\mu_1$  be mutually absolutely continuous measures on  $\Omega$ . Let  $W \sim \frac{1}{2}\mu_0 + \frac{1}{2}\mu_1$ , and let  $Z = \frac{d\mu_1}{d\mu_0}(W)$ . We assume that  $\mu_0, \mu_1$  are such that the distribution of  $Z$  is non-atomic.*

Here  $Z$  can be interpreted as the likelihood ratio of the events that  $W$  came from either  $\mu_0$  or  $\mu_1$ . Notice that in particular our assumption implies that  $\mu_0 \neq \mu_1$ , since otherwise  $Z$  is equal to one with probability one.

**Definition 2.2.** *Let  $V$  be a countable set of agents, which we take to equal  $\{1, 2, \dots, n\}$  in the finite case and  $\mathbb{N} = \{1, 2, \dots\}$  in the infinite case.*

*Let  $\{0, 1\}$  be the set of possible values of the **state of the world**  $S$ , and let  $\delta_0$  and  $\delta_1$  be the distributions on  $\{0, 1\}$  such that  $\delta_0(0) = \delta_1(1) = 1$ .*

*Let  $W_i \in \Omega$  be agent  $i$ 's **private signal**, and denote  $\bar{W} = (W_1, W_2, \dots)$ .*

*Let*

$$(S, \bar{W}) \sim \mathbb{P},$$

*where*

$$\mathbb{P} = \mathbb{P}_{\mu_0, \mu_1, V} = \left( \frac{1}{2}\delta_0\mu_0^V + \frac{1}{2}\delta_1\mu_1^V \right).$$

Equivalently,  $\mathbb{P}[S = 1] = \mathbb{P}[S = 0] = 1/2$ , and, conditioned on  $S$ ,  $W_i$  are i.i.d.  $\mu_S$ . We shall henceforth use  $\mathbb{P}[\cdot]$  and  $\mathbb{E}[\cdot]$  to denote probabilities and expectations in the distribution described above. We will implicitly extend this probability space when we allow mixed strategies.

**Definition 2.3.** *Agent  $i$ 's **private belief**  $I_i$  is defined by*

$$I_i = \mathbb{P}[S = 1 | W_i].$$

Note that  $I_i$  is well defined since  $\mu_0$  and  $\mu_1$  are mutually absolutely continuous, and that by the condition in Definition 2.1  $I_i$  it has a non-atomic distribution.

**Definition 2.4.** A **social network**  $G = (V, E)$  is a directed graph. We assume throughout that it is simple and strongly connected. Let the set of neighbors of  $i \in V$  be  $\partial i = \{j : (i, j) \in E\} \cup \{i\}$  (i.e.,  $\partial i$  includes  $i$ ). The **out-degree** of  $i$  is equal to  $|\partial i|$ , and is assumed to always be finite.

Finite out-degrees mean that an agent observes the actions of a finite number of other agents. We do allow infinite **in-degrees**; this corresponds to agents whose actions are observed by infinitely many other agents.

We consider the discrete time periods  $t = 0, 1, 2, \dots$ , where in each period each agent  $i \in V$  has to choose one of the actions in  $\{0, 1\}$ . This action is a function of agent  $i$ 's private belief, as well as the actions of its neighbors in previous time periods, and so can be thought of as a function from  $[0, 1] \times \{0, 1\}^{|\partial i| \cdot t}$  to  $\{0, 1\}$ .

**Definition 2.5.** A **pure strategy at time  $t$**  of an agent  $i \in V$  is a Borel-measurable function  $q_t^i : [0, 1] \times \{0, 1\}^{|\partial i| \cdot t} \rightarrow \{0, 1\}$ . A **pure strategy** of an agent  $i$  is the sequence of functions  $q^i = (q_0^i, q_1^i, \dots)$ , where  $q_t^i$  is  $i$ 's pure strategy at time  $t$ . A **mixed strategy**  $Q^i$  of agent  $i$  is a pure-strategy-valued random variable. We shall henceforth refer to mixed strategies simply as **strategies**.

A **strategy profile** is the set of (mixed) strategies  $\bar{Q} = \{Q^i : i \in V\}$ .

Note that strategy profiles are in general mixed, and so  $\mathbb{P}$  will denote averaging over this additional randomness. Each random variable  $Q^i$  will be taken to be independent of all other  $Q^j$ , as well as all the other previously defined random variables.

**Definition 2.6.** Fix a social network  $G = (V, E)$  and a strategy profile  $\bar{Q}$ . The **action** of agent  $i$  at time  $t$  is denoted by  $A_t^i \in \{0, 1\}$ . Denote the **history** of actions of the neighbors of  $i$  before time  $t$  by  $A_{[0,t)}^{\partial i} = \{A_{t'}^j : t' < t, j \in \partial i\}$ . The actions are recursively defined as follows:

$$A_t^i = A_t^i(G, \bar{Q}) = Q_t^i(I_i, A_{[0,t)}^{\partial i}).$$

Note that we limit the action to be a function of the private belief  $I_i$ , as opposed to the private signal  $W_i$ . However, as will become apparent when we next define the agents' utilities, a strategy that discriminates between private signals that induce identical private beliefs can always be replaced by an equivalent strategy that does not.

**Definition 2.7.** Let  $0 < \lambda < 1$  denote the agents' common **discount factor**. Given a social network  $G$  and strategy profile  $\bar{Q}$ , agent  $i$ 's **utility at time  $t$** ,  $u_{i,t}$ , is given by

$$u_{i,t} = u_{i,t}(G, \bar{Q}) = \mathbb{P} [A_t^i(G, \bar{Q}) = S].$$

Agent  $i$ 's **utility**  $u_i$  is given by

$$u_i = u_i(G, \lambda, \bar{Q}) = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t u_{i,t}(G, \bar{Q}).$$

Note that  $u_i \in [0, 1]$ , due to the normalization factor  $(1 - \lambda)$ .

**Definition 2.8.** A **game**  $\mathcal{G}$  is a 4-tuple  $(\mu_0, \mu_1, \lambda, G)$  consisting of two measures, a discount factor and a social network graph, satisfying the conditions of the definitions above. The agents' strategies and utilities are those of Definitions 2.5 and 2.7.

**Definition 2.9.** Given a game  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$ , strategy profile  $\bar{Q}$  is an **equilibrium** if, for every agent  $i \in V$  it holds that

$$u_i(\bar{Q}) \geq u_i(\bar{R}),$$

for any  $\bar{R}$  such that  $R^j = Q^j$  for all  $j \neq i$  in  $V$ .

**Definition 2.10.** Fix a game  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  and a strategy profile  $\bar{Q}$ .

Let  $C_i$  be the set of actions (i.e., a subset of  $\{0, 1\}$ ) that  $i$  takes infinitely often:

$$C_i = C_i(G, \bar{Q}) = \{s \in \{0, 1\} : A_t^i(G, \bar{Q}) = s \text{ for infinitely many values of } t\}.$$

We call  $C_i$  the **infinite action set** of  $i$ . When  $C_i = C_j$  for all  $i, j$  then we denote  $C = C_i = C_j$ .

We show below in Theorem 3 that there always exists a random variable  $C = C(G, \bar{Q})$  such that almost surely  $C_i = C$ , for all agents  $i$ .

**Definition 2.11.** Let  $G = (V, E)$  be a directed graph. A (directed) **path** of length  $k$  from  $i \in V$  to  $j \in V$  in  $G$  is sequence of  $k + 1$  nodes  $i_1, \dots, i_{k+1}$  such that  $(i_n, i_{n+1}) \in E$  for  $n = 1, \dots, k$ , and where  $i_1 = i$  and  $i_{k+1} = j$ .

**Definition 2.12.** A directed graph  $G$  is  **$L$ -connected** if, for each  $(i, j) \in E$ , there exists a path of length at most  $L$  in  $G$  from  $j$  to  $i$ .

Equivalently,  $G$  is  $L$ -connected if whenever there exists a path of length  $k$  from  $i$  to  $j$ , there exists a path of length at most  $L \cdot k$  from  $j$  back to  $i$ . Note that 1-connected graphs are commonly known as undirected graphs.

## 2.2 Results

Our first result is a preliminary lemma that shows that an equilibrium always exists. This result is not completely straightforward, especially in the case of infinite graphs.

**Lemma 2.13.** Every game  $\mathcal{G}$  has an equilibrium.

The next theorem is our main result.

**Theorem 1** (Learning). Fix  $\mu_0, \mu_1$ , a discount factor  $\lambda \in (0, 1)$ , and positive integers  $L$  and  $d$ . Let  $G$  be an infinite,  $L$ -connected degree  $d$  graph, let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  be a game, and let  $\bar{Q}$  be any equilibrium strategy profile of  $\mathcal{G}$ . Then almost surely

$$C_i = \{S\}$$

for all  $i \in V$ .

That is, all agents learn  $S$  and converge to the correct action. We also prove a version of Theorem 1 for finite graphs.

**Theorem 2** (Learning in finite graphs). Fix  $\mu_0, \mu_1$ , a discount factor  $\lambda \in (0, 1)$ , and positive integers  $L$  and  $d$ . Then there exists a sequence  $q(n) = q(\mu_0, \mu_1, L, d, \lambda, n)$  such that  $\lim_n q(n) = 1$  and the following holds. Let  $G$  be an  $L$ -connected graph of degree at most  $d$  with at least  $n$  vertices, let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  be a game, and let  $\bar{Q}$  be an equilibrium strategy profile of  $\mathcal{G}$ . Then

$$\mathbb{P}[C_i = \{S\} \text{ for all } i \in V] \geq q(n).$$

An important component of the learning theorems above is the following agreement theorem, which stands as an independent result.

**Theorem 3** (Agreement). *Let  $\mathcal{G}$  be a game, and let  $\bar{Q}$  be an equilibrium strategy profile of  $\mathcal{G}$ . Then there exists a random variable  $C$  such that with probability one  $C_i = C$  for all agents  $i \in V$ .*

A result in [20] states that when likelihood ratios are unbounded (i.e., the convex closure of the support of the private beliefs is  $[0, 1]$ ) and agreement is reached (as described in Theorem 3) then learning occurs in any game with a binary state of the world and conditionally i.i.d. private signals. Hence any cases of non-learning (necessarily in non-egalitarian networks, by Theorems 1 and 2 above) must feature bounded likelihood ratios.

## 3 Proofs

### 3.1 Rooted graphs

In the proofs that follow we will make repeated use of the notion of a *rooted graph*. This section starts with some basic definitions and culminates in the definition of a topology on rooted graphs. In this we follow our previous work [19] which builds on the work of others such as Benjamini and Schramm [6] and Aldous and Steele [2].

**Definition 3.1.** A **rooted graph** is a pair  $(G, i)$ , where  $G = (V, E)$  is a directed graph, and  $i \in V$  is a vertex in  $G$ .

**Definition 3.2.** Let  $G = (V, E)$  and  $G' = (V', E')$  be directed graphs, and let  $(G, i)$  and  $(G', i')$  be rooted graphs. A **rooted graph isomorphism** between  $(G, i)$  and  $(G', i')$  is a bijection  $h : V \rightarrow V'$  such that

1.  $h(i) = i'$ .
2.  $(j, k) \in E \Leftrightarrow (h(j), h(k)) \in E'$ .

If  $(G, i)$  and  $(G', i')$  are such that there exists a graph isomorphism between them then we say that they are isomorphic and write  $(G, i) \cong (G', i')$ . Informally, isomorphic graphs cannot be told apart when vertex labels are removed; equivalently, one can be turned into the other by an appropriate renaming of the vertices. The isomorphism class of  $(G, i)$  is the set of rooted graphs that are isomorphic to it, and will be denoted by  $[G, i]$ .

**Definition 3.3.** Let  $i, j$  be vertices in a directed graph  $G$ . Denote by  $\Delta(i, j)$  the length of the shortest directed path from  $i$  to  $j$ .

Note that in general  $\Delta(i, j) \neq \Delta(j, i)$ , since the graph is directed.

**Definition 3.4.** The (directed) **ball**  $B_r(G, i)$  of radius  $r$  of the rooted graph  $(G, i)$  is the rooted graph, with root  $i$ , induced in  $G$  by the set of vertices  $\{j \in V : \Delta(i, j) \leq r\}$ .

Let  $[G', i']$  and  $[G, i]$  be isomorphism classes of strongly connected rooted graphs. The distance  $D([G, i], [G', i'])$  is defined by

$$D([G, i], [G', i']) = \inf\{2^{-r} : B_r(G, i) \cong B_r(G', i')\}. \quad (1)$$

It is straightforward to show that this is indeed a well defined metric; a diagonalization argument is needed to show that when  $D([G, i], [G', i']) = 0$  then  $(G, i) \cong (G', i')$ . Note also that while it is

not crucial that the graphs are strongly connected, it is crucial that there is a path from the root to each vertex.

Intuitively, the larger the radius around  $i$  in which the graphs are isomorphic, the closer they are. In fact, the quantitative dependence of  $D(\cdot, \cdot)$  on  $r$  (exponential in our definition) will not be of importance here, and we shall only be interested in the topology induced by this metric. We will henceforth refer to this topology when discussing rooted graphs.

**Claim 3.5.** *Let  $\{[G_n, i_n]\}_{n=1}^\infty$  be a sequence of rooted graph isomorphism classes such that*

$$\lim_{n \rightarrow \infty} [G_n, i_n] = [G, i].$$

*Then for every  $r > 0$  there exists an  $N > 0$  such that for all  $n > N$  it holds that  $B_r(G_n, i_n) \cong B_r(G, i)$ . Furthermore, there exists a subsequence  $\{[G_{n_r}, i_{n_r}]\}_{r=1}^\infty$  such that  $B_r(G_{n_r}, i_{n_r}) \cong B_r(G, i)$ .*

*Proof.* The first part of the claim follows directly from the definition of limits and Eq. 1. The second part holds for  $n_r = \min\{n : B_r(G_n, i_n) \cong B_r(G, i)\}$ , which is guaranteed to be finite by the first part.  $\square$

### 3.2 Locality

An important observation is that the actions and the utility of agent, up to *time*  $t$ , depends only on the strategies of the agents that are at *distance*  $t$  from it. We formalize this notion in this section.

**Claim 3.6.** *Let  $\mathcal{G}_1 = (\mu_0, \mu_1, \lambda, G_1)$  and  $\mathcal{G}(\mu_0, \mu_1, \lambda, G_2)$  be games. Let  $h$  be a rooted graph isomorphism between  $B_{r+1}(G_1, i_1)$  and  $B_{r+1}(G_2, i_2)$  for some  $r > 0$ , and let  $Q_1^{j_1} = Q_2^{j_2}$  for all  $j_1 \in B_r(G_1, i_1)$  and  $j_2 = h(j_1)$ .*

*Then the games, as probability spaces, can be coupled so that  $A_t^{i_1} = A_t^{i_2}$  for all  $t \leq r$ .*

Some care needs to be taken with a statement such as “agent  $j_1$  plays the same strategy as agent  $j_2$ ”; it can only be meaningful in the context of a bijection that identifies each neighbor of  $j_1$  with each neighbor of  $j_2$ . We here naturally take this bijection to be  $h$ , and accordingly demand that it be an isomorphism between balls of radius  $r + 1$  (rather than  $r$ ), so that the neighbors of the agents on the surface of the ball are also mapped.

*Proof.* Couple the two processes by equating the states of the world and setting  $W_{j_1} = W_{h(j_1)}$  for all  $j_1 \in B_r(G_1, i_1)$ , and furthermore coupling the choices of pure strategies of  $j_1$  and  $h(j_1)$ .

We shall prove by induction a stronger statement, namely that under the claim hypothesis,  $A_t^{j_1} = A_t^{j_2}$  for any  $j_1 \in B_r(G_1, i_1)$ ,  $j_2 = h(j_1)$  and  $t \leq r - \Delta(i_1, j_1)$ .

We prove the statement by induction on  $t$ . For  $t = 0$ ,  $A_0^{j_1}$  depends only on agent  $j_1$ ’s private signal and choice of pure strategy, which are both equal to those of  $j_2$ . Hence  $A_0^{j_1} = A_0^{j_2}$  for all  $j \in B_r(G_1, i_1)$ .

Assume now that the claim holds up to some  $t - 1 \leq r - 1$ . Let  $j_1$  be such that  $t \leq r - \Delta(i_1, j_1)$ . We would like to show that  $A_t^{j_1} = A_t^{j_2}$ . Let  $k_1$  be a neighbor of  $j_1$ . Then  $t - 1 \leq r - \Delta(i_1, k_1)$ , and so  $A_{t'}^{k_1} = A_{t'}^{k_2}$  for all  $t' \leq t - 1$ , by the inductive assumption. Since  $A_t^{j_1}$  depends only on  $j_1$ ’s private signals, choice of pure strategy and the actions of its neighbors in previous time periods, and since these are all identical to those of  $j_2$ , then it indeed follows that  $A_t^{j_1} = A_t^{j_2}$ .  $\square$

Recalling the definition

$$u_{i,t} = \mathbb{P} [A_t^i = S],$$

the following corollary is a direct consequence of this claim.

**Corollary 3.7.** *Let  $\mathcal{G}_1 = (\mu_0, \mu_1, \lambda, G_1)$  and  $\mathcal{G}(\mu_0, \mu_1, \lambda, G_2)$  be games. Let  $h$  be a rooted graph isomorphism between  $B_{r+1}(G_1, i_1)$  and  $B_{r+1}(G_2, i_2)$  for some  $r > 0$ , and let  $Q_1^{j_1} = Q_2^{j_2}$  for all  $j_1 \in B_r(G_1, i_1)$  and  $j_2 = h(j_1)$ .*

*Then  $u_{i_1, t} = u_{i_2, t}$  for all  $t \leq r$ .*

### 3.3 A topology on strategies and the existence of equilibria

In the following theorem we show that the agents' set of strategies admits a compact topology which preserves the continuity of the utilities. From this we infer the existence of equilibria for this game, which is not immediate, since the number of players may be infinite. We also use this topology to define a compact topology on equilibria, which is a component of the proof of our main theorem.

For a fixed private belief  $I_i$ , a pure strategy is a function from the actions of neighbors to actions, which we call a response.

**Definition 3.8.** *Let  $G = (V, E)$  be a social network. A **response at time  $t$**  of an agent  $i \in V$  is a function  $r_{i,t} : \{0, 1\}^{|\partial i| \cdot t} \rightarrow \{0, 1\}$ . A **response** of an agent  $i$  is the sequence of functions  $r_i = (r_{i,0}, r_{i,1}, \dots)$ . Let  $\mathcal{R}_i$  be the space of responses of agent  $i$ .*

A (mixed) strategy of agent  $i$  can be thought of as a measure on the product space  $[0, 1] \times \mathcal{R}_i$  of private beliefs and responses, with the marginal on the first coordinate equaling the distribution of  $I_i$ . Milgrom and Weber [18] call this representation a *distributional strategy*. They show that for a Bayesian game with a finite number of players, and given some conditions, the weak topology on distributional strategies is compact and keeps the utilities continuous. Hence, by Glicksberg's theorem [13], the game has an equilibrium. The next theorem shows that these conditions apply in our case, when the number of agents is finite.

**Lemma 3.9.** *Fix  $G = (V, E)$ , with  $V$  finite. Then for each agent  $i$  there exists a topology  $\mathcal{T}_i$  on its strategy space such that the strategy space is compact and the utilities  $u_j$  are continuous in the product of the strategy spaces.*

*Proof.* We prove by showing that the conditions of Theorem 1 in [18] are met.

1. The set of private beliefs (*types* in the language of [18]) is  $[0, 1]$ , a complete separable metric space, as required. Furthermore, the distribution of private beliefs is absolutely continuous with respect to the product of their marginal distributions. This fulfills condition R2 of [18].
2. The utilities  $u_j$  are bounded, measurable functions of the private beliefs and the responses.
3. Define a metric  $D$  on  $i$ 's responses  $\mathcal{R}_i$  by

$$D(r_i, r'_i) = \exp(-\min\{t : r_{i,t} \neq r'_{i,t}\}).$$

This can be easily verified to indeed be a metric. By a standard diagonalization argument it follows that  $\mathcal{R}_i$  is compact in the topology induced by this metric, as required.

Furthermore, for fixed private beliefs, the utilities  $u_j$  are equicontinuous in the responses: if a response is changed by at most  $\delta = e^{-T}$  (in terms of the metric  $D$ ) then it remains unchanged in the first  $T$  time periods, and so the utilities are changed by at most  $\epsilon = (1 - \lambda) \sum_{t=T}^{\infty} \lambda^t = \lambda^T$ . This fulfills condition R1 of [18].

Since these conditions are met, it follows by Theorem 1 in [18] that the mixed strategies of agent  $i$  are compact in the weak topology  $\mathcal{T}_i$ , and that the utilities  $u_j$  are, under  $\mathcal{T}_i$ , a continuous function of the strategies.  $\square$



Note that under the above defined topology on  $\mathcal{R}_i$  the set of pure strategies is separable, and so the topology  $\mathcal{T}_i$  on (mixed) strategies is metrizable, e.g. with the Lévy-Prokhorov metric [8].

Given Lemma 3.9, a direct application of Glicksberg's theorem [13] yields that every game *with a finite number of agents* has an equilibrium. The extension of this result to infinitely many agents is not immediate, and requires us to invoke the graphical nature of the game. We do this in the following section. Before turning to that we will prove an additional version of Lemma 3.9.

**Lemma 3.10.** *Fix  $G = (V, E)$ , with  $V$  finite. Then for each agent  $i$  there exists a topology  $\mathcal{T}_i$  on its strategy space such that the strategy space is compact and the utilities in each time  $t$ ,  $u_{j,t}$ , are continuous in the product of the strategy spaces.*

*Proof.* The proof is identical to the proof of Lemma 3.9 above, except that we let each agent's utility be defined by  $u'_j = u_{j,t}$ ; that is, we set the discount factor to be one at time  $t$  and zero otherwise. Since in the proof above we required of the discount factors nothing more than to have a finite sum, the proof still applies, and the utilities (in this case  $u_{j,t}$ ), are continuous in the product of the strategy spaces.  $\square$

### 3.4 The space of rooted graph strategies and its topology

Lemma 3.9 implies, by Glicksberg's fixed point theorem [13], that any game on a finite graph has an equilibrium. However, Glicksberg's theorem only applies to a finite number of agents, and therefore we cannot use it to prove that equilibria exist in general. To that end we define we define a topology on *rooted graph strategies*.

**Definition 3.11.** *Let  $G = (V, E)$  and  $G' = (V', E')$  be strongly connected directed graphs. Let  $(G, i)$  and  $(G', i')$  be rooted graphs. Let  $\bar{Q}$  and  $\bar{R}$  be strategy profiles for the agents in  $V$  and  $V'$ , respectively. We say that the triplet  $(G, i, \bar{Q})$  is equivalent to the triplet  $(G', i', \bar{R})$  if there exists a rooted graph isomorphism  $h$  from  $(G, i)$  to  $(G', i')$  such that  $\bar{Q}^j = \bar{R}^{h(j)}$  for all  $j \in V$ . The space of **rooted graph strategies**  $\mathcal{GS}$  is the set of equivalence classes induced by this equivalence relation. We denote an element of this space by  $[G, i, \bar{Q}]$ .*

Recall that we defined above a metrizable compact topology  $\mathcal{T}_i$  on agent  $i$ 's strategy; let  $d$  be a metric that induces  $\mathcal{T}_i$ . Let  $i$  and  $i'$  be agents in graphs  $G$  and  $G'$ , respectively. We can use  $d$  as a metric between their strategies, as long as we uniquely identify each neighbor of one with a neighbor of the other. Let  $h$  be a bijection between  $\partial i'$  and  $\partial i$ . Then  $d_h(Q^i, Q^{i'})$  will denote the distance thus defined between  $Q^i$  and  $Q^{i'}$ .

We now use the metrics of strategies and of rooted graphs to define a metric on rooted graph strategies.

**Definition 3.12.** *Let  $[G, i, \bar{Q}]$  and  $[G', i', \bar{R}]$  be rooted graph strategies. For  $r \in \mathbb{N}$ , let  $H(r)$  be the (perhaps empty) set of rooted graph isomorphisms between  $B_r(G, i)$  and  $B_r(G', i')$ . Define the distance  $D([G, i, \bar{Q}], [G', i', \bar{R}])$  by*

$$D([G, i, \bar{Q}], [G', i', \bar{R}]) = \inf_{r \in \mathbb{N}} \left\{ \max \left\{ 2^{-r}, \min_{h \in H(r+1)} \max_{j \in B_r(G, i)} d_h(Q^j, \bar{R}^{h(j)}) \right\} \right\}. \quad (2)$$

Note that the choice of  $h \in H(r+1)$  and then  $j \in B_r(G, i)$  guarantees that  $h$  is a bijection from the set of neighbors of  $j$  to the set of neighbors of  $h(j)$ . It is straightforward (if tedious) to show that  $D(\cdot, \cdot)$  is indeed a well defined metric.

**Definition 3.13.** The utility map  $u : \mathcal{GS} \rightarrow \mathbb{R}$  is given by

$$u([G, i, \bar{Q}]) = u_i(G, \bar{Q}).$$

The utility map at time  $t$   $u_t : \mathcal{GS} \rightarrow \mathbb{R}$  is given by

$$u_t([G, i, \bar{Q}]) = u_{i,t}(G, \bar{Q}).$$

This is a straightforward recasting of Definition 2.7 into the language of rooted graph strategy spaces. Note that the map is well defined, as clearly  $u_i(G, \bar{Q}) = u_{i'}(G', \bar{R})$  if  $(G, i, \bar{Q}) \cong (G', i', \bar{R})$ ; the two are simply different namings of a group of agents with an identical social network and identical strategies, who therefore have identical utilities.

**Lemma 3.14.** The utility map  $u : \mathcal{GS} \rightarrow \mathbb{R}$  is continuous.

Equivalently, if  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$  then  $u_{i_n}(G_n, \bar{Q}_n) \rightarrow_n u_i(G, \bar{Q})$ .

*Proof.* We will prove the claim by showing that  $u_t$  is continuous. The claim will follow because, by the bounded convergence theorem, if  $f$  is a linear combination of the uniformly bounded maps  $\{f_t\}_{t=0}^\infty$ , with summable positive coefficients, then the continuity of all the maps  $f_t$  implies the continuity of  $f$ .

Let  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$ . We will show that  $u_t([G_n, i_n, \bar{Q}_n]) \rightarrow_n u_t(G, i, \bar{Q})$ .

Consider a sequence of games  $\mathcal{G}_n$  which are all played on the finite graph  $B = B_{t+1}(G, i)$ . Since  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$  then there exists some  $N$  such that, for  $n > N$ , it holds that  $D([G_n, i_n, \bar{Q}_n], [G, i, \bar{Q}]) < 2^{-(t+1)}$ . Hence, by the definition of  $D(\cdot, \cdot)$ , it holds that  $B \cong B_{t+1}(G_n, i_n)$  for  $n > N$ . Denote by  $h_n$  an isomorphism between the two balls that minimizes  $\max_{j \in B_t(G, i)} d_{h_n}(Q^j, Q_n^{h_n(j)})$ , as appears in the definition of  $D(\cdot, \cdot)$ .

Let each agent  $j$  in  $B_t(G, i)$  play  $Q_n^{h_n(j)}$  in  $\mathcal{G}_n$ , and let the rest of the agents in  $B$  (i.e., those at distance  $t+1$  from  $i$ ) play arbitrary strategies. Denote by  $\bar{R}_n$  the strategy profile played by the agents at game  $\mathcal{G}_n$ , and denote by  $\bar{R}$  the restriction of  $\bar{Q}$  to  $B$ . By Corollary 3.7,  $u_t([G_n, i_n, \bar{Q}_n]) = u_t([B, i, \bar{R}_n])$  and  $u_t([G, i, \bar{Q}]) = u_t([B, i, \bar{R}])$ . Therefore it is left to show that  $u_t([B, i, \bar{R}_n]) \rightarrow_n u_t([B, i, \bar{R}])$ .

Now,  $D([G_n, i_n, \bar{Q}_n], [G, i, \bar{Q}]) \rightarrow 0$ . Hence  $d(R^j, R_n^j) \rightarrow 0$ , and so the strategies of each agent  $j$  in  $B_t(G, i)$  converge to the strategy  $R^j = Q^j$ . Furthermore, the strategies in  $B_t(G, i)$  converge uniformly, since there is only a finite number of them, and so the strategy profile converges in the product topology. It follows that  $u_{i_n, t}$ , which by Lemma 3.10 is a continuous function of the strategies in  $B_t(G, i)$ , converges to  $u_{i, t}$ .  $\square$

**Claim 3.15.** Let  $\{[G_n, i_n, \bar{Q}_n]\}_{n=1}^\infty$  be a sequence of graph strategies such that

1.  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$ .
2.  $\bar{Q}_n$  is an equilibrium strategy profile of  $\mathcal{G}_n = (\mu_0, \mu_1, \lambda, G_n)$ .

Then  $\bar{Q}$  is an equilibrium strategy profile of  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$ .

*Proof.* Let  $\bar{R}$  be a strategy profile for the agents in  $G$  such that  $\bar{R}^j = \bar{Q}^j$  for all  $j \neq i$ . We will show that  $u_i(G, \bar{Q}) \geq u_i(G, \bar{R})$ .

Let  $\bar{R}_n$  be the strategy profile for agents on  $G_n$  defined by  $\bar{R}_n^j = \bar{Q}_n^{j_n}$  for  $j_n \neq i_n$ , and let  $\bar{R}_n^{i_n} = \bar{R}^i$ . Note that  $[G_n, i_n, \bar{R}_n] \rightarrow_n [G, i, \bar{R}]$ .

Since  $\bar{Q}_n$  is an equilibrium profile of  $\mathcal{G}_n$ ,

$$u_{i_n}(G, \bar{Q}_n) \geq u_{i_n}(G, \bar{R}_n).$$

Taking the limit of both sides and substituting the definition of the utility map we get that

$$\lim_{n \rightarrow \infty} u([G_n, i_n, \bar{Q}_n]) \geq \lim_{n \rightarrow \infty} u(G_n, i_n, \bar{R}_n).$$

Finally, since by Lemma 3.14 above the utility map is continuous, we have that

$$u([G, i, \bar{Q}]) \geq u([G, i, \bar{R}]).$$

□

**Claim 3.16.** *Let  $\{[G_n, i_n, \bar{Q}_n]\}_{n=1}^\infty$  be a sequence of rooted graph strategies such that the limit  $\lim_n [G_n, i_n]$  exists and is equal to some  $[G, i]$ . Then there exists a subsequence  $\{[G_{n_k}, i_{n_k}, \bar{Q}_{n_k}]\}_{k=1}^\infty$  such that the limit  $\lim_k [G_{n_k}, i_{n_k}, \bar{Q}_{n_k}]$  exists and is equal to  $[G, i, \bar{Q}]$  for some strategy profile  $\bar{Q}$ .*

*Proof.* By Claim 3.5 there exists a subsequence  $\{[G_{n_r}, i_{n_r}]\}_{r=1}^\infty$  such that  $B_r(G_{n_r}, i_{n_r}) \cong B_r(G, i)$ . We will therefore assume without loss of generality that  $n_r = n$ , i.e., limit ourselves to this subsequence. Accordingly, let  $h_n : V \rightarrow V_n$  be a sequence rooted graph isomorphisms between  $B_n(G, i)$  and  $B_n(G_n, i_n)$ . Note that since out-degrees are finite then  $B_n(G, i)$  is finite for all  $n$ .

Let  $j$  be a vertex of  $G$ , and let  $r_j$  be the graph distance between  $i$  and  $j$ . For  $n \geq r_j + 1$ , denote  $j_n = h_n(j)$ . Note that  $h_n$  also maps the neighbors of  $j_n$  to the neighbors of  $j$ .

We will now construct  $\bar{Q}$ , the strategy profile of the agents in  $G$  such that  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$ , starting with agent 1 of  $G$ . Since  $\mathcal{T}_1$  is compact, the sequence  $\{Q_n^{1n}\}_{n=r_1+1}^\infty$  has a converging subsequence, i.e., one along which  $d_{h_n}(Q_n^{1n}, Q^1) \rightarrow_n 0$  for some strategy  $Q^1$ , which we will assign to agent 1 in  $G$ . Likewise, this subsequence has a subsequence along which  $d_{h_n}(Q_n^{2n}, Q^2) \rightarrow_n 0$ , etc. Thus, by a standard diagonalization argument, we have that there exists a subsequence  $\{[G_{n_k}, i_{n_k}, \bar{Q}_{n_k}]\}_{k=1}^\infty$  with isomorphisms  $h_{n_k}$  such that  $d_{h_{n_k}}(Q_{n_k}^{j_{n_k}}, Q^j) \rightarrow_k 0$  for all  $j$ . It is now straightforward to verify that  $D([G_{n_k}, i_{n_k}, \bar{Q}_{n_k}], [G, i, \bar{Q}]) \rightarrow_k 0$ : pick some  $r > 0$  and then  $k$  large enough so that  $h_{n_k}$  is an isomorphism between  $B_{r+1}(G_n, i_n)$  and  $B_{r+1}(G, i)$ . Then by definition (Eq. 2)

$$D([G_{n_k}, i_{n_k}, \bar{Q}_{n_k}], [G, i, \bar{Q}]) \leq \max \left\{ 2^{-r}, \max_{j \in B_r(G, i)} d_{h_{n_k}}(Q_{n_k}^{j_{n_k}}, Q^j) \right\}.$$

If we now further increase  $k$  then  $d_{h_{n_k}}(Q_{n_k}^{j_{n_k}}, Q^j) \rightarrow_k 0$ , and since  $B_r(G, i)$  is finite then we have that  $D([G_{n_k}, i_{n_k}, \bar{Q}_{n_k}], [G, i, \bar{Q}]) \leq 2^{-r}$ , for  $k$  large enough. Since this holds for all  $r$  then

$$D([G_{n_k}, i_{n_k}, \bar{Q}_{n_k}], [G, i, \bar{Q}]) \rightarrow_k 0$$

and

$$\lim_{k \rightarrow \infty} [G_{n_k}, i_{n_k}, \bar{Q}_{n_k}] = [G, i, \bar{Q}].$$

□

**Theorem 3.17** (2.13). *Every game  $\mathcal{G}$  has an equilibrium.*

*Proof.* Let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$ . Let  $i$  be a vertex in  $G$ , denote  $G_n = B_n(G, i)$ , and denote its root by  $i_n$ . Let  $\{\mathcal{G}_n = (\mu_0, \mu_1, \lambda, G_n)\}_{n=1}^\infty$  be a sequence of finite games with equilibria strategy profiles  $\bar{Q}_n$ . Then  $[G_n, i_n] \rightarrow_n [G, i]$ , and so by Claim 3.16 we have that there exists a strategy profile  $\bar{Q}$  and a subsequence  $\{[G_{n_k}, i_{n_k}]\}_{k=1}^\infty$  such that

$$\lim_{k \rightarrow \infty} [G_{n_k}, i_{n_k}, \bar{Q}_{n_k}] = [G, i, \bar{Q}].$$

Finally, by Claim 3.15,  $\bar{Q}$  is an equilibrium profile of  $\mathcal{G}$ .  $\square$

### 3.5 Compact graphs and graph strategies

**Definition 3.18.** Let  $\mathcal{E}(L, d)$  be the space of isomorphism classes of  $L$ -connected, degree  $d$ , strongly connected rooted graphs, equipped with the topology induced by the metric  $D(\cdot, \cdot)$ .

That is,  $\mathcal{E}(L, d)$  is a space of **egalitarian** graphs. The following is a standard result (see [19]).

**Theorem 3.19.**  $\mathcal{E}(L, d)$  is compact.

Since our topology on rooted graph isomorphism classes is metrizable, compactness implies that  $\mathcal{E}(L, d)$  is also sequentially compact, i.e., that every sequence in  $\mathcal{E}(L, d)$  has a converging subsequence, converging to a point in  $\mathcal{E}(L, d)$ .

**Definition 3.20.** Let  $\mathcal{K}$  be a set of graphs. Denote by  $\mathcal{R}(\mathcal{K})$  the set of rooted graph isomorphism classes  $[G, i]$  such that  $G \in \mathcal{K}$ .

Let  $\mathcal{R}$  be a set of rooted graph isomorphism classes. Denote by  $\mathcal{GS}(\mathcal{R})$  the set of rooted graph strategies  $[G, i, \bar{Q}]$  such that  $[G, i] \in \mathcal{R}$  and  $\bar{Q}$  is an equilibrium strategy profile for  $G$ .

**Claim 3.21.** Let  $\mathcal{R}$  be a compact set of rooted graph isomorphism classes. Then  $\mathcal{GS}(\mathcal{R})$  is a compact set of equilibrium rooted graph strategies.

*Proof.* Since our topology on graph strategies is metrizable, we will prove the claim by showing that any sequence of points in  $\mathcal{GS}(\mathcal{R})$  has a converging subsequence with a limit in  $\mathcal{GS}(\mathcal{R})$ .

Let  $\{[G_n, i_n, \bar{Q}_n]\}_{n=1}^\infty$  be a sequence of points in  $\mathcal{GS}(\mathcal{R})$ . Since  $\mathcal{R}$  is compact, the sequence  $\{[G_n, i_n]\}_{n=1}^\infty$  has a converging subsequence  $\{[G_{n_k}, i_{n_k}]\}_{k=1}^\infty$  that converges to some  $[G, i] \in \mathcal{R}$ . Hence, by Claim 3.16, the sequence  $\{[G_n, i_n, \bar{Q}_n]\}_{k=1}^\infty$  has a converging subsequence that, for some  $\bar{Q}$ , converges to  $[G, i, \bar{Q}]$ . Finally, by Claim 3.15  $\bar{Q}$  is an equilibrium strategy profile for  $G$ , and so  $[G, i, \bar{Q}] \in \mathcal{GS}(\mathcal{R})$ .  $\square$

### 3.6 Agreement

Recall that the *infinite action set*  $C_i$  of agent  $i$  is defined by

$$C_i = C_i(G, \bar{Q}) = \{s \in \{0, 1\} : A_t^i(G, \bar{Q}) = s \text{ for infinitely many values of } t\}.$$

We shall show in this section that it follows from the work of Rosenberg, Solan and Vieille [24] that any action in  $C_i$  is also an optimal action, as far as agent  $i$  can tell; every agent stops behaving strategically at some point. This will be useful in the proof of Theorem 3, which states that  $C_i = C_j$  for all  $i, j$ .

Recall that a strategy of agent  $i$  at time  $t$  is a function of its private belief  $I_i$  and the actions of its neighbors in previous time periods,  $A_{[0, t)}^i$ . Hence we can think of the sigma-algebra generated by these random variables as the “information available to agent  $i$  at time  $t$ ”:

**Definition 3.22.** Denote by

$$\mathcal{F}_t^i = \mathcal{F}_t^i(G, \bar{Q}) = \sigma(I_i, Q^i, A_{[0,t)}^{\partial i})$$

the information available to agent  $i$  at time  $t$ , and denote by

$$\mathcal{F}_\infty^i = \mathcal{F}_\infty^i(G, \bar{Q}) = \sigma(\cup_{t=0}^\infty \mathcal{F}_t^i)$$

the information available to agent  $i$  at the limit  $t \rightarrow \infty$ .

Note that  $\mathcal{F}_t^i$  includes the sigma-algebra generated by  $i$ 's private belief, the actions of  $i$ 's neighbors before time  $t$ , and  $i$ 's pure strategy;  $i$  knows which pure strategy it has chosen.

Since the utility of action  $s$  at time  $t$  is  $\mathbb{P}[s = S]$ , a myopic agent would take an action  $s$  in  $\{0, 1\}$  that maximizes  $\mathbb{P}[s = S | \mathcal{F}_t^i]$ . This motivates the following definition:

**Definition 3.23.** Denote by

$$B_t^i = B_t^i(G, \bar{Q}) = \operatorname{argmax}_{s \in \{0,1\}} \mathbb{P}[s = S | \mathcal{F}_t^i(G, \bar{Q})]$$

the best response of agent  $i$  at time  $t$ . Likewise denote by

$$B_\infty^i = B_\infty^i(G, \bar{Q}) = \operatorname{argmax}_{s \in \{0,1\}} \mathbb{P}[s = S | \mathcal{F}_\infty^i]$$

the set of best responses of agent  $i$  at the limit  $t \rightarrow \infty$ .

At any time  $t$  there is indeed almost surely only one action that maximizes  $\mathbb{P}[s = S | \mathcal{F}_t^i(G, \bar{Q})]$ , since we require that the distribution of private beliefs be non atomic. This does not hold at the limit  $t \rightarrow \infty$ , and so we let  $B_\infty^i$  be a set which can take the values  $\{0\}$ ,  $\{1\}$  or  $\{0, 1\}$ . Note that  $B_\infty^i = \{0, 1\}$  iff  $\mathbb{P}[S = 1 | \mathcal{F}_\infty^i] = \frac{1}{2}$ .

The following theorem is a restatement, in our notation, of Proposition 2.1 in Rosenberg, Solan and Vieille [24].

**Theorem 3.24** (Rosenberg, Solan and Vieille). *Let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  be a game with equilibrium  $\bar{Q}$ . Then for any agent  $i$  it holds that  $C_i(G, \bar{Q}) \subseteq B_\infty^i(G, \bar{Q})$  almost surely.*

That is, any action that  $i$  takes infinitely often is optimal, given all the information agent  $i$  eventually learns. Note that this theorem is stated in [24] for a finite number of agents. However, a careful reading of the proof reveals that it holds equally for a countably infinite set of agents. The same holds for their Theorem 2.3, in which they further prove the following agreement result.

**Theorem 3.25** (Rosenberg, Solan and Vieille). *Let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  be a game with equilibrium  $\bar{Q}$ . Let  $j$  be a neighbor of  $i$ . Then  $C_j(G, \bar{Q}) \subseteq B_\infty^i(G, \bar{Q})$  almost surely.*

Equivalently, if  $i$  observes  $j$ , and  $j$  takes an action  $a$  infinitely often, then  $a$  is an optimal action for  $i$ . If we could show that  $B_\infty^i = C_i$  for all  $i$ , it would follow from these two theorems, and the fact that the graph is strongly connected, that  $C_i = C_j$  for all agents  $i$  and  $j$ ; the agents would agree on their optimal action sets. This is precisely what we shall do in the remainder of this section.

**Definition 3.26.** Denote by  $Z_t^i$  the log-likelihood ratio of the events  $S = 1$  and  $S = 0$ , conditioned on  $\mathcal{F}_t^i$ , the information available to agent  $i$  at time  $t$

$$Z_t^i = \log \frac{\mathbb{P}[S = 1 | \mathcal{F}_t^i]}{\mathbb{P}[S = 0 | \mathcal{F}_t^i]},$$

and let

$$Z_\infty^i = \log \frac{\mathbb{P}[S = 1 | \mathcal{F}_t^i]}{\mathbb{P}[S = 0 | \mathcal{F}_t^i]}.$$

Let  $Y_t^i$  be defined as follows:

$$Y_t^i = \log \frac{\mathbb{P}[A_{[0,t)}^{\partial i} | A_{[0,t)}^i, Q^i, S = 1]}{\mathbb{P}[A_{[0,t)}^{\partial i} | A_{[0,t)}^i, Q^i, S = 0]},$$

where  $A_{[0,t)}^i$  is the sequence of actions of  $i$  up to time  $t - 1$ , and  $A_{[0,t)}^{\partial i}$  is the sequence of actions of  $i$ 's neighbors up to time  $t - 1$ . Finally, let

$$Y_\infty^i = \lim_{t \rightarrow \infty} Y_t^i.$$

Note that it is not clear that the limit  $\lim_t Y_t^i$  exists. We show this in the following claim.

**Claim 3.27.** Denote by  $I_{-i}$  the private beliefs of all agents but  $i$ . Then

1.  $\lim_t Z_t^i = Z_\infty^i$  almost surely.
2.  $Z_t^i = Y_t^i + Z_0^i$ .
3.  $\lim_t Y_t^i = Y_\infty^i$  almost surely, and  $Y_\infty^i$  is measurable in  $\sigma(A_{[0,\infty)}^i, I_{-i}, \bar{Q})$ .

*Proof.* 1. Recall that

$$Z_t^i = \log \frac{\mathbb{P}[S = 1 | \mathcal{F}_t^i]}{\mathbb{P}[S = 0 | \mathcal{F}_t^i]}.$$

Since  $\{\mathcal{F}_t^i\}_{t=0}^\infty$  is a filtration then  $\mathbb{P}[S = 1 | \mathcal{F}_t^i]$  is a martingale, which converges a.s. since it is bounded. Hence  $Z_t^i$  also converges, and in particular

$$\lim_{t \rightarrow \infty} Z_t^i = \log \frac{\mathbb{P}[S = 1 | \mathcal{F}_\infty^i]}{\mathbb{P}[S = 0 | \mathcal{F}_\infty^i]} = Z_\infty^i.$$

2. By the definition of  $\mathcal{F}_t^i$

$$\begin{aligned} Z_t^i &= \log \frac{\mathbb{P}[S = 1 | I_i, Q^i, A_{[0,t)}^{\partial i}]}{\mathbb{P}[S = 0 | I_i, Q^i, A_{[0,t)}^{\partial i}]} \\ &= \log \frac{\mathbb{P}[A_{[0,t)}^{\partial i} | I_i, Q^i, S = 1]}{\mathbb{P}[A_{[0,t)}^{\partial i} | I_i, Q^i, S = 0]} \frac{\mathbb{P}[S = 1 | I_i, Q^i]}{\mathbb{P}[S = 0 | I_i, Q^i]}, \end{aligned}$$

where the second equality follows from Bayes' law. Now, conditioned on  $S$  and  $i$ 's pure strategy  $Q^i$ , the probability for a sequence of actions  $A_{[0,t)}^{\partial i}$  of  $i$ 's neighbors depends on  $I_i$  only in as much as  $I_i$  affects  $i$ 's actions up to time  $t - 1$ ,  $A_{[0,t)}^i$ . Hence  $\mathbb{P}[A_{[0,t)}^{\partial i} | I_i, Q^i, S] = \mathbb{P}[A_{[0,t)}^{\partial i} | A_{[0,t)}^i, Q^i, S]$ . Note also that

$$\frac{\mathbb{P}[S = 1 | I_i, Q^i]}{\mathbb{P}[S = 0 | I_i, Q^i]} = Z_0^i.$$

Therefore

$$Z_t^i = Y_t^i + Z_0^i.$$

3. Since  $Z_t^i$  converges almost surely and  $Z_t^i = Y_t^i + Z_0^i$  then  $Y_t^i$  also converges almost surely. Since each  $Y_t^i$  is a function of  $A_{[0,t]}^{\partial^i}$  and  $Q^i$ , it follows that their limit,  $Y_\infty^i$ , is measurable in  $\sigma(A_{[0,\infty)}^{\partial^i}, Q^i)$ . However, given  $\bar{Q}$ ,  $A_{[0,\infty)}^{\partial^i}$  is a function of  $I_{-i}$  and  $A_{[0,\infty)}^i$ : for a choice of pure strategies the actions of all agents but  $i$  can be determined given their private signals and the actions of  $i$ . Hence  $Y_\infty^i$  is also measurable in  $\sigma(A_{[0,\infty)}^i, I_{-i}, \bar{Q})$ .  $\square$

**Claim 3.28.** *The distribution of  $Z_0^i$  is non-atomic, as is the distribution of  $Z_0^i$  conditioned on  $S$ .*

*Proof.* By definition,

$$Z_0^i = \log \frac{\mathbb{P}[S = 1 | I_i, Q^i]}{\mathbb{P}[S = 0 | I_i, Q^i]}.$$

However, the choice of strategy  $Q^i$  is independent of both  $I_i$  and  $S$ , and so

$$Z_0^i = \log \frac{\mathbb{P}[S = 1 | I_i]}{\mathbb{P}[S = 0 | I_i]} = \log \frac{I_i}{1 - I_i}.$$

Since the distribution of  $I_i$  is non-atomic (see Definition 2.3 and the comment after it) then so is the distribution of  $Z_0^i$ . Since  $S$  takes only two values then the same holds when conditioned on  $S$ .  $\square$

**Theorem 3.29.** *Let  $\mathcal{G} = (\mu_0, \mu_1, \lambda, G)$  be a game with equilibrium  $\bar{Q}$ . Then almost surely  $B_\infty^i(G, \bar{Q}) = C_i(G, \bar{Q})$ .*

*Proof.* By its definition,  $C_i$  takes values in  $\{\{0\}, \{1\}, \{0, 1\}\}$ , and by Theorem 3.24 we have that  $C_i \subseteq B_\infty^i$ . Therefore the claim holds when  $B_\infty^i = \{0\}$  or  $B_\infty^i = \{1\}$ , and it remains to show that  $C_i = \{0, 1\}$  when  $B_\infty^i = \{0, 1\}$ , or that  $\mathbb{P}[C_i \neq \{0, 1\}, B_\infty^i = \{0, 1\}] = 0$ .

Let  $a = (a_0, a_1, \dots)$  be a sequence of actions, each in  $\{0, 1\}$ , in which only one action appears infinitely often. Since there are only countably many such sequences, then if  $\mathbb{P}[C_i \neq \{0, 1\}, B_\infty^i = \{0, 1\}] > 0$ , then there exists such a sequence  $a$  for which  $\mathbb{P}[A_{[0,\infty)}^i = a, B_\infty^i = \{0, 1\}] > 0$ . We shall prove the claim by showing that  $\mathbb{P}[A_{[0,\infty)}^i = a, B_\infty^i = \{0, 1\}] = 0$ .

Recall that by Claim 3.27, the event that  $B_\infty^i = \{0, 1\}$  is equal to the event that  $Z_0^i = -Y_\infty^i$ . Recall also that by the same claim,  $Y_\infty^i$  is measurable in  $\sigma(A_{[0,\infty)}^i, I_{-i}, \bar{Q})$ . Hence

$$\begin{aligned} \mathbb{P}[A_{[0,\infty)}^i = a, B_\infty^i = \{0, 1\} | S, I_{-i}, \bar{Q}] &= \mathbb{P}[A_{[0,\infty)}^i = a, Z_0^i = -Y_\infty^i(a, I_{-i}, \bar{Q}) | S, I_{-i}, \bar{Q}] \\ &\leq \mathbb{P}[Z_0^i = -Y_\infty^i(a, I_{-i}, \bar{Q}) | S, I_{-i}, \bar{Q}] \end{aligned}$$

Now, by Claim 3.28,  $Z_0^i$  conditioned on  $S$  has a non-atomic distribution. Further conditioning on  $\bar{Q}$  and  $I_{-i}$  leaves its distribution unchanged, since it is independent of the former, and independent of the latter conditioned on  $S$ . Hence the probability that it equals  $-Y_\infty^i(a, S, I_{-i}, \bar{Q})$  is zero. Hence

$$\mathbb{P}[A_{[0,\infty)}^i = a, Z_0^i] = \mathbb{E}[\mathbb{P}[A_{[0,\infty)}^i = a, Z_0^i = -Y_\infty^i | S, I_{-i}, \bar{Q}]] = 0.$$

$\square$

Our agreement theorem is a direct consequence of Theorem 3.29.

*Proof of Theorem 3.* Let  $i$  and  $j$  be agents. Since  $G$  is strongly connected, there exists a path from  $i$  to  $j$ . By Theorem 3.25 we have, by induction along this path, that  $C_j \subseteq B_\infty^i$  almost surely. But  $C_i = B_\infty^i$  by Theorem 3.29 above, and so we have that  $C_j \subseteq C_i$ . However, there also exists a path from  $j$  back to  $i$ , and so  $C_i \subseteq C_j$ , and the two are equal. This holds for any pair of agents, and so it follows that there exists a random variable  $C$  such that  $C_i = C$  for all  $i$ , almost surely.  $\square$

### 3.7 $\delta$ -independence

In this section we define a notion of near independence between two random variables, and state some related lemmas. The following definition is also taken (almost) verbatim from [19].

Let  $d_{TV}(\cdot, \cdot)$  denote the total variation distance between two measures defined on the same measurable space. Let  $A$  and  $B$  be two random variables with joint distribution  $\mu_{(A,B)}$ . Then we denote by  $\mu_A$  the marginal distribution of  $A$ ,  $\mu_B$  the marginal distribution of  $B$ , and  $\mu_A \times \mu_B$  the product distribution of the marginal distributions.

**Definition 3.30.** Let  $(X_1, X_2, \dots, X_k)$  be random variables. We refer to them as  **$\delta$ -independent** if

$$d_{TV}(\mu_{(X_1, \dots, X_k)}, \mu_{X_1} \times \dots \times \mu_{X_k}) \leq \delta.$$

I.e., the joint distribution  $\mu_{(X_1, \dots, X_k)}$  has total variation distance of at most  $\delta$  from the product of the marginal distributions  $\mu_{X_1} \times \dots \times \mu_{X_k}$ . Likewise,  $(X_1, \dots, X_l)$  are  **$\delta$ -dependent** if the total variation distance between these distributions is more than  $\delta$ . We shall also make use of the following notation:

**Definition 3.31.** Let  $X_1, \dots, X_k$  be random variables, and let  $S$  be a binary random variable. We say that  $(X_1, \dots, X_k)$  are  **$\delta$ -independent conditioned on  $S$**  if they are  $\delta$ -independent conditioned on both  $S = 0$  and  $S = 1$ . Denote

$$\text{dep}_S(X_1, \dots, X_k) = \min\{\delta : (X_1, \dots, X_k) \text{ are } \delta\text{-independent conditioned on } S\}$$

Note that this minimum is indeed attained, by the definition of  $\delta$ -independence.

The proofs of the following two general claims are elementary and fairly straightforward. They appear in [19].

**Claim 3.32.** Let  $A$ ,  $B$  and  $C$  be random variables such that  $\mathbb{P}[A \neq B] \leq \delta$  and  $(B, C)$  are  $\delta'$ -independent. Then  $(A, C)$  are  $2\delta + \delta'$ -independent.

**Claim 3.33.** Let  $A = (A_1, \dots, A_k)$ , and  $X$  be random variables. Let  $(A_1, \dots, A_k)$  be  $\delta_1$ -independent and let  $(A, X)$  be  $\delta_2$ -independent. Then  $(A_1, \dots, A_k, X)$  are  $(\delta_1 + \delta_2)$ -independent.

**Definition 3.34.** Let  $S$  be a binary random variable such that  $\mathbb{P}[S = 1] = 1/2$ . We say that the binary random variables  $(X_1, \dots, X_k)$  are  $(p, \delta)$ -good estimators of  $S$  if the following hold:

1.  $\mathbb{P}[X_\ell = S] \geq p$ , for  $\ell = 1, \dots, k$ .
2.  $(X_1, \dots, X_k)$  are  $\delta$ -independent, conditioned on  $S$ .

The following lemma captures the idea that sufficiently many  $(p, \delta)$ -good estimators give arbitrarily good estimates, for any  $p > \frac{1}{2}$  and  $\delta$  small enough.



**Claim 3.35.** *Let  $S$  be a binary random variable such that  $\mathbb{P}[S = 1] = 1/2$ , and let  $(X_1, \dots, X_k)$  be  $(\frac{1}{2} + \epsilon, \delta)$ -good estimators of  $S$ .*

*Then there exists a function  $a : \{0, 1\}^k \rightarrow \{0, 1\}$  such that*

$$\mathbb{P}[a(X_1, \dots, X_k) = S] > 1 - e^{-2\epsilon^2 k} - \delta.$$

*Proof.* Let  $(Y_1, \dots, Y_k)$  be random variables such that the distribution of  $(S, Y_i)$  is equal to the distribution of  $(S, X_i)$  for all  $i$ , and let  $(Y_1, \dots, Y_k)$  be independent, conditioned on  $S$ . Then  $(X_1, \dots, X_k)$  can be coupled to  $(Y_1, \dots, Y_k)$  in such a way that they differ only with probability  $\delta$ . Therefore, if we show that  $\mathbb{P}[a(Y_1, \dots, Y_k) = S] > q + \delta$  for some  $a$  then it will follow that  $\mathbb{P}[a(X_1, \dots, X_k) = S] > q$ .

Denote  $\hat{Y} = \frac{1}{k} \sum_{i=1}^k Y_i$ , and denote  $\alpha_0 = \mathbb{E}[\hat{Y} | S = 0]$  and  $\alpha_1 = \mathbb{E}[\hat{Y} | S = 1]$ . It follows that

$$\alpha_1 - \alpha_0 = \frac{1}{k} \sum_{i=1}^k (2\mathbb{P}[Y_i = S] - 1) > 2\epsilon.$$

By the Hoeffding bound

$$\mathbb{P}[\hat{Y} \leq \alpha_1 - \epsilon | S = 1] < e^{-2\epsilon^2 k}$$

and

$$\mathbb{P}[\hat{Y} \geq \alpha_0 + \epsilon | S = 0] < e^{-2\epsilon^2 k}.$$

Let  $a(Y_1, \dots, Y_k) = \mathbf{1}_{\hat{Y} > \alpha_1 - \epsilon}$ . Then by the above we have that  $\mathbb{P}[a(Y_1, \dots, Y_k) \neq S] < e^{-2\epsilon^2 k}$ , and so

$$\mathbb{P}[a(X_1, \dots, X_k) = S] > 1 - e^{-2\epsilon^2 k} - \delta.$$

□

### 3.8 The probability of learning

We say that an agent “learns  $S$ ” if  $S$  is the only action that it takes infinitely often, i.e., if  $C_i = \{S\}$ . By Theorem 3  $C_i = C_j$ , and so if one agent learns  $S$  then all learn. We now start to explore the probability of learning, with the ultimate goal of proving that it equals one under certain conditions.

**Definition 3.36.** *Let the probability of learning map  $p : \mathcal{GS} \rightarrow \mathbb{R}$  be given by*

$$p([G, i, \bar{Q}]) = \lim_{t \rightarrow \infty} \mathbb{P}[A_{i,t}(G, \bar{Q}) = S].$$

Before showing that  $p$  is well defined (i.e., the limit exists), and proving that it is lower semi-continuous, we make the following additional definition.

**Definition 3.37.** *Let*

$$\hat{S}_\infty = \hat{S}_\infty([G, i, \bar{Q}]) = \begin{cases} 0 & B_\infty^i(G, \bar{Q}) = \{0\} \\ 1 & B_\infty^i(G, \bar{Q}) = \{1\} \\ 1 & B_\infty^i(G, \bar{Q}) = \{0, 1\} \end{cases}$$

*be a maximum a posteriori (MAP) estimator of  $S$  given  $\mathcal{F}_\infty^i$ , for some agent  $i$  in  $G$ .*

Note that  $\hat{S}_\infty([G, i, \bar{Q}])$  is independent of  $i$ , since, by Theorem 3,  $B_\infty^i = C$  for all agents  $i, j$  in  $G$ . Note also that  $\hat{S}_\infty$  is indeed a MAP estimator of  $S$  given  $\mathcal{F}_\infty^i$ , since by definition  $B_\infty^i$  is the set of most probable estimates of  $S$ , given  $\mathcal{F}_\infty^i$ .

**Claim 3.38.**

$$p([G, i, \bar{Q}]) = \mathbb{P} \left[ \hat{S}_\infty([G, i, \bar{Q}]) = S \right].$$

It follows that  $p$  is well defined.

*Proof.* Recall that  $C = B_\infty^i$  by Theorem 3.29. Therefore

$$\lim_{t \rightarrow \infty} \mathbb{P} [A_{i,t} = S | C = \{0, 1\}] = \lim_{t \rightarrow \infty} \mathbb{P} [A_{i,t} = S | B_\infty^i = \{0, 1\}].$$

Since the event that  $B_\infty^i = \{0, 1\}$  is identical to the event that  $\mathbb{P} [S = 1 | \mathcal{F}_\infty^i] = \frac{1}{2}$ , and since  $A_{i,t}$  is  $\mathcal{F}_\infty^i$ -measurable for all  $t$ , then it follows that

$$\lim_{t \rightarrow \infty} \mathbb{P} [A_{i,t} = S | C = \{0, 1\}] = \frac{1}{2}.$$

and also that

$$\lim_{t \rightarrow \infty} \mathbb{P} [\hat{S}_\infty = S | C = \{0, 1\}] = \frac{1}{2}.$$

When  $C = \{0\}$  or  $C = \{1\}$  then  $\lim_t A_{i,t} = \hat{S}_\infty$ , and so

$$\lim_{t \rightarrow \infty} \mathbb{P} [A_{i,t} = S | C \neq \{0, 1\}] = \mathbb{P} [\hat{S}_\infty = S | C \neq \{0, 1\}].$$

Since we have equality when conditioning on both  $C \neq \{0, 1\}$  and  $C = \{0, 1\}$  then we also have unconditioned equality and

$$p([G, i, \bar{Q}]) = \lim_{t \rightarrow \infty} \mathbb{P} [A_{i,t} = S] = \mathbb{P} [\hat{S}_\infty = S].$$

□

Since  $\hat{S}_\infty([G, i, \bar{Q}])$  is independent of  $i$  then the following is a direct consequence of Claim 3.38.

**Corollary 3.39.**  $p([G, i, \bar{Q}]) = p([G, j, \bar{Q}])$  for all  $i, j$ .

This corollary also follows from a claim that appears in Rosenberg, Solan and Vielle [24].

**Definition 3.40.** Given  $\mu_0$  and  $\mu_1$ , denote  $p^*(\mu_0, \mu_1) = \frac{1}{2} + \frac{1}{2}d_{TV}(\mu_0, \mu_1)$ .

**Claim 3.41.** Given  $\mu_0$  and  $\mu_1$

$$p([G, i, \bar{Q}]) \geq p^*(\mu_0, \mu_1) > \frac{1}{2}$$

for any  $G, i$  and equilibrium strategy profile  $\bar{Q}$ .

*Proof.*  $p^*(\mu_0, \mu_1) > \frac{1}{2}$ , since  $\mu_0 \neq \mu_1$ . Let  $\hat{S}_{i,0}$  be the maximum a posteriori (MAP) estimator of  $S$  given  $i$ 's private signal,  $W_i$ . Then (see Claim 3.30 in [19])

$$\mathbb{P} [\hat{S}_{i,0} = S] = p^*(\mu_0, \mu_1).$$

Now,  $\hat{S}_\infty$  is a MAP estimator of  $S$  given  $\mathcal{F}_\infty^i$ . Since  $\mathcal{F}_\infty^i$  includes  $W_i$  then

$$\mathbb{P} [\hat{S}_\infty = S] \geq \mathbb{P} [\hat{S}_{i,0} = S] = p^*(\mu_0, \mu_1),$$

and the claim follows by Claim 3.38. □

**Theorem 3.42.**  *$p$  is lower semi-continuous, i.e., if  $[G_n, i_n, \bar{Q}_n] \rightarrow_n [G, i, \bar{Q}]$  then  $\liminf_n p([G_n, i_n, \bar{Q}_n]) \geq p([G, i, \bar{Q}])$ .*

*Proof.* Recall that the utility of agent  $i$  at time  $t$  is given by the utility map at time  $t$  (Definition 3.13):

$$u_t([G, i, \bar{Q}]) = \mathbb{P} [A_t^i(G, \bar{Q}) = S] .$$

Hence an alternative definition of  $p$  is that

$$p([G, i, \bar{Q}]) = \lim_{t \rightarrow \infty} u_t([G, i, \bar{Q}]) .$$

Now,  $A_t^i$  is  $\mathcal{F}_\infty^i$ -measurable. Hence, since  $\hat{S}_\infty$  is a MAP estimator of  $S$  given  $\mathcal{F}_\infty^i$ , it follows that

$$\mathbb{P} [\hat{S}_\infty = S] \geq \mathbb{P} [A_t^i = S] ,$$

or that

$$p([G, i, \bar{Q}]) \geq u_t([G, i, \bar{Q}]) .$$

Hence

$$p([G, i, \bar{Q}]) = \sup_t u_t([G, i, \bar{Q}]) ,$$

and since  $u_t$  is continuous (see the proof of Lemma 3.14), it follows that  $p$  is lower semi-continuous.  $\square$

### 3.9 Proof of main theorems

The following lemma, which we shall prove in the next section, is the main ingredient in the proof of our main Theorem 1. Before stating it we would like to remind the reader that if  $\mathcal{K}$  is a set of graphs then  $\mathcal{R}(\mathcal{K})$  is the set of rooted  $\mathcal{K}$  graphs, and  $\mathcal{GS}(\mathcal{R}(\mathcal{K}))$  is the set of  $\mathcal{K}$  equilibrium graph strategies (Definition 3.20). Recall also that  $p^*(\mu_0, \mu_1) = \frac{1}{2} + \frac{1}{2}d_{TV}(\mu_0, \mu_1)$  (Definition 3.40).

**Lemma 3.43.** *Let  $G$  be an infinite, strongly connected graph such that the closure of  $\mathcal{GS}(\mathcal{R}(\{G\}))$  is compact. Then for all equilibrium strategy profiles on  $G$ ,  $k \in \mathbb{N}$ ,  $\epsilon > 0$  and  $\delta > 0$  there exists an agent  $i$  in  $G$ , a time  $t$  and  $\mathcal{F}_t^i$ -measurable binary random variables  $(X_1, \dots, X_k)$  that are  $(p^*(\mu_0, \mu_1) - \epsilon, \delta)$ -good estimators of  $S$ .*

The next theorem is a more general version of our main Theorem 1. We shall prove it now, assuming Lemma 3.43, and derive from it the proof of Theorem 1.

**Theorem 3.44.** *Let  $G$  be an infinite, strongly connected graph such that the closure of  $\mathcal{GS}(\mathcal{R}(\{G\}))$  is compact, and let  $\bar{Q}$  be any equilibrium strategy profile of  $\mathcal{G}$ . Then*

$$\mathbb{P} [C(G, \bar{Q}) = \{S\}] = 1 .$$

*Proof.* We first show that  $p([G, i, \bar{Q}]) = 1$ . Assume by way of contradiction that  $p < 1$ . By Claim 3.41, we have that  $p > 1/2$ .

By Lemma 3.43, for every  $k \in \mathbb{N}$ , and  $\delta > 0$  there exist  $(X_1, \dots, X_k)$  that are  $\mathcal{F}_t^i$ -measurable for some  $i$  and  $t$ , are  $\delta$ -independent conditioned on  $S$ , and are each equal to  $S$  with probability greater than one half, since  $p^*(\mu_0, \mu_1) > \frac{1}{2}$ .

By Claim 3.35 it follows that for  $k$  large enough and  $\delta$  small enough, there exists an estimator  $\hat{S}$  of  $S$  that is a function of  $(X_1, \dots, X_k)$ , and is equal to  $S$  with probability strictly greater than  $p$ .

This  $\hat{S}$  is  $\mathcal{F}_\infty^i$ -measurable, and so a MAP estimator of  $S$  given  $\mathcal{F}_\infty^i$  must also equal  $S$  with probability greater than  $p$ . However,  $\hat{S}_\infty$  in a MAP estimator of  $S$  given  $\mathcal{F}_\infty^i$ , and it equals  $S$  with probability  $p$  (Claim 3.38), and so we have a contradiction. Hence  $p([G, i, \bar{Q}]) = 1$ .

Now, by Claim 3.38 we have that

$$\mathbb{P} \left[ \hat{S}_\infty([G, i, \bar{Q}]) = S \right] = p([G, i, \bar{Q}]) = 1.$$

By the definition of  $\hat{S}_\infty$  we have that  $\hat{S}_\infty = \{S\}$  iff  $B_\infty^i = \{S\}$  for some (equivalently all)  $i$ . Since, by Theorem 3,  $C = C_i = B_\infty^i$ , it follows that  $\mathbb{P}[C = \{S\}] = 1$ .  $\square$

The proofs of our main theorems now follow.

*Proof of Theorem 1.* Since  $G$  is  $L$ -connected and of degree  $d$ , then so are all the graphs in  $\mathcal{R}(\{G\})$ . By Claim 3.19,  $\mathcal{E}(L, d)$ , the set of  $L$ -connected degree  $d$  rooted graphs, is compact. Hence the closure of  $\mathcal{R}(\{G\}) \subseteq \mathcal{E}(L, d)$  is also compact, and so by Claim 3.21 we have that  $\mathcal{GS}(\mathcal{R}(\{G\}))$  is compact. The theorem therefore follows from Theorem 3.44 above.  $\square$

*Proof of Theorem 2.* Let  $\mathcal{K}_n$  be the set of  $L$ -connected, degree  $d$  graphs with  $n$  vertices. Since  $\mathcal{K}_n$  is finite then  $\mathcal{R}(\mathcal{K}_n)$  is compact, and so, by Claim 3.21,  $\mathcal{GS}(\mathcal{K}_n)$  is also compact. Since the map  $p$  is lower semi-continuous (Theorem 3.42), then it attains a minimum on  $\mathcal{GS}(\mathcal{K}_n)$ . Let  $[G_n, i_n, \bar{Q}_n]$  be a minimum point, and denote  $q(n) = p([G_n, i_n, \bar{Q}_n])$ .

By Theorem 3.19, the set of  $L$ -connected, degree  $d$  graphs is compact, and so, by again invoking Claim 3.21, we have that the sequence  $\{[G_n, i_n, \bar{Q}_n]\}_{n=1}^\infty$  has a converging subsequence that must converge to some *infinite*  $L$ -connected, degree  $d$  equilibrium graph strategy  $[G, i, \bar{Q}]$ . By Theorem 1, we have that  $p([G, i, \bar{Q}]) = 1$ , and so, by lower semi-continuity of  $p$ , it follows that

$$\lim_{n \rightarrow \infty} q(n) = \lim_{n \rightarrow \infty} p([G_n, i_n, \bar{Q}_n]) \geq p([G, i, \bar{Q}]) = 1.$$

$\square$

### 3.10 Proof of Lemma 3.43

Before proving Lemma 3.43 we prove the following claim.

**Claim 3.45.** *Let  $[G, i_0, \bar{Q}]$  be an equilibrium graph strategy. Let  $\{i_n\}_{n=1}^\infty$  be a sequence of vertices such that the graph distance  $\Delta(i_0, i_n)$  diverges with  $n$ . Fix  $t$ , and for each  $n$  let  $X^{i_n}$  be  $\mathcal{F}_t^{i_n}$ -measurable. Then*

$$\lim_{n \rightarrow \infty} \text{dep}_S(X^{i_n}, \hat{S}_\infty) = 0.$$

*Proof.* Let  $\mathcal{B}_r^i = \sigma(\{W_j, Q^j : j \in B_r(G, i)\})$ . We first show, by induction on  $r$ , that  $\mathcal{F}_r^i \subseteq \mathcal{B}_r^i$ : any  $\mathcal{F}_r^i$ -measurable random variable is also  $\mathcal{B}_r^i$ -measurable. It will follow that  $X^{i_n}$  is  $\mathcal{B}_t^{i_n}$ -measurable.

Clearly  $\mathcal{F}_0^i \subseteq \mathcal{B}_0^i$ . Assume now that  $\mathcal{F}_{r'}^j \subseteq \mathcal{B}_{r'}^j$  for all  $j$  and  $r' < r$ . By definition,  $\mathcal{F}_r^i = \sigma(\mathcal{F}_{r-1}^i, A_{r-1}^{i_0})$ . For  $j \in \partial i$  we have that  $A_{r-1}^j$  is  $\mathcal{B}_{r-1}^j$ -measurable. Finally,  $\mathcal{B}_{r-1}^j \subseteq \mathcal{B}_r^i$ , and so  $\mathcal{F}_r^i \subseteq \mathcal{B}_r^i$ .

Note that for  $i, j$  and  $r_1, r_2$  such that  $B_{r_1}(G, i)$  and  $B_{r_2}(G, j)$  are disjoint it holds that  $\mathcal{B}_{r_1}^i$  and  $\mathcal{B}_{r_2}^j$  are independent conditioned on  $S$ , since the choices of pure strategies are independent and private beliefs are independent conditioned on  $S$ .

Let  $R_r^i$  be a MAP estimator of  $\hat{S}_\infty$  given  $\mathcal{B}_r^i$ . Since  $\Delta(i_0, i_n) \rightarrow_n \infty$ , it follows that for any  $r$  and  $n$  large enough  $B_t(G, i_n)$  and  $B_r(G, i_0)$  are disjoint, and so  $X^{i_n}$  and  $R_r^{i_0}$  are independent, conditioned on  $S$ . For such  $n$ , by Claim 3.32, we have that  $(X^{i_n}, \hat{S}_\infty)$  are  $2\delta$ -independent, for  $\delta = \mathbb{P}[R_r^i \neq \hat{S}_\infty]$ .

Finally, since  $\hat{S}_\infty$  is  $\mathcal{B}_\infty^i$ -measurable, it follows that

$$\lim_{r \rightarrow \infty} \mathbb{P}[R_r^i \neq \hat{S}_\infty] = 0,$$

and so

$$\lim_{n \rightarrow \infty} \text{dep}_S(X^{i_n}, \hat{S}_\infty) = 0.$$

□

*Proof of Lemma 3.43.* Denote by  $\mathcal{C}$  the closure of  $\mathcal{GS}(\mathcal{R}(\{G\}))$ . Note that by Claim 3.15 any graph strategy in  $\mathcal{C}$  is in equilibrium.

We shall prove by induction a stronger claim, namely that under the claim hypothesis, for all  $[H, j, \bar{Q}] \in \mathcal{C}$ ,  $k \in \mathbb{N}$ ,  $\epsilon > 0$  and  $\delta > 0$  there exists an agent  $i$  in  $G$ , a time  $t$  and  $\mathcal{F}_t^i$ -measurable binary random variables  $(X_1, \dots, X_k)$  that are  $(\epsilon, \delta)$ -good estimators of  $S$ .

We prove the claim by induction on  $k$ . The claim holds trivially for  $k = 0$ . Assume that the claim holds up to  $k$ .

Let  $[H, j, \bar{Q}] \in \mathcal{C}$ . Let  $\{j_n\}_{n=1}^\infty$  be a sequence of vertices in  $H$  such that  $\lim_n \Delta(j, j_n) = \infty$ . Since  $\mathcal{C}$  is compact then there exists a converging sequence  $[H, j_n, \bar{Q}] \rightarrow_n [F, i', \bar{R}] \in \mathcal{C}$ . By the inductive assumption, there exists an agent  $i$  in  $F$ , a time  $t$  and random variables  $(X_1^i, \dots, X_k^i)$  which are  $\mathcal{F}_t^i$ -measurable and are  $(\epsilon', \delta')$ -good estimators of  $S$ , for some  $0 < \epsilon' < \epsilon$  and  $0 < \delta' < \delta$ . Denote

$$\bar{X}_k^i = (X_1^i, \dots, X_k^i).$$

Let  $r = \Delta(i', i)$ . Since for  $n$  large enough  $B_r(H, j_n) \cong B_r(F, i')$  then, if we let  $i_n \in B_r(H, j_n)$  be the vertex that corresponds to  $i \in B_r(F, i')$  then  $[H, i_n, \bar{R}] \rightarrow_n [F, i, \bar{R}]$ , and  $\lim_n \Delta(j, i_n) = \infty$ .

Since  $\bar{X}_k^i$  is  $\mathcal{F}_t^i$ -measurable, there exists a function  $x_k^i$  such that  $\bar{X}_k^i = x_k^i(I_i, A_{[0,t]}^{\partial i})$ . Denote  $\bar{X}_k^{i_n} = x_k^i(I_{i_n}, A_{[0,t]}^{\partial i_n})$ .

Now, since the strategies of agents in the neighborhood of  $i_n$  in  $H$  converge in the weak topology to those of  $i$  in  $F$ , then the random variables  $\{(S, \bar{X}_k^{i_n})\}_{n=1}^\infty$  converge in the weak topology to  $(S, \bar{X}_k^i)$ . Moreover, the measures of these random variables are over the finite space  $\{0, 1\}^{k+1}$ , and so we also have convergence in total variation. In particular,  $(X_1^{i_n}, \dots, X_k^{i_n})$  approach  $\delta'$ -independence:

$$\lim_{n \rightarrow \infty} \text{dep}_S(X_1^{i_n}, \dots, X_k^{i_n}) = \text{dep}_S(X_1^i, \dots, X_k^i) \leq \delta'. \quad (3)$$

Likewise,

$$\lim_{n \rightarrow \infty} \mathbb{P}[X_\ell^{i_n} = S] = \mathbb{P}[X_\ell^i = S] > p^*(\mu_0, \mu_1) - \epsilon'. \quad (4)$$

for  $\ell = 1, \dots, k$ . Additionally, since  $\Delta(j, i_n) \rightarrow_n \infty$ , it follows by Claim 3.45 that

$$\lim_{n \rightarrow \infty} \text{dep}_S(\bar{X}_k^{i_n}, \hat{S}_\infty) = 0, \quad (5)$$

that is,  $\bar{X}_k^{i_n}$  and  $\hat{S}_\infty$  are practically independent, for large  $n$ .

Now, recall that  $\hat{S}_\infty$  is  $\mathcal{F}_\infty^i$ -measurable. Therefore, if we let  $R_{t'}^{i_n}$  be a MAP estimator of  $\hat{S}_\infty$  given  $\mathcal{F}_{t'}^i$  then for any  $n$  it holds that

$$\lim_{t' \rightarrow \infty} \mathbb{P} \left[ R_{t'}^{i_n} = \hat{S}_\infty \right] = 1. \quad (6)$$

By Claim 3.32, a consequence of Eqs. 5 and 6 is that

$$\lim_{n \rightarrow \infty} \lim_{t' \rightarrow \infty} \text{dep}_S(\bar{X}_k^{i_n}, R_{t'}^{i_n}) = 0.$$

That is,  $\bar{X}_k^{i_n}$  and  $R_{t'}^{i_n}$  are practically independent, for large enough  $n$  and  $t'$ . It follows by Claim 3.33 that

$$\lim_{n \rightarrow \infty} \lim_{t' \rightarrow \infty} \text{dep}_S(X_1^{i_n}, \dots, X_k^{i_n}, R_{t'}^{i_n}) = \delta'. \quad (7)$$

It follows from Eq. 6 that

$$\lim_{t' \rightarrow \infty} \mathbb{P} \left[ R_{t'}^{i_n} = S \right] = \mathbb{P} \left[ \hat{S}_\infty = S \right] \geq p^*(\mu_0, \mu_1). \quad (8)$$

Gathering the above results, we have that for  $n$  and  $t'$  large enough,

1.  $\mathbb{P} \left[ X_\ell^{i_n} = S \right] \geq p^*(\mu_0, \mu_1) - \epsilon$ , by Eq. 4. Likewise  $\mathbb{P} \left[ R_{t'}^{i_n} = S \right] = \mathbb{P} \left[ \hat{S}_\infty = S \right] \geq p^*(\mu_0, \mu_1) - \epsilon$  by Eq. 8.
2.  $(X_1^{i_n}, \dots, X_k^{i_n}, R_{t'}^{i_n})$  are  $\delta$ -independent, by Eq. 7.

We therefore have that  $(X_1^{i_n}, \dots, X_k^{i_n}, R_{t'}^{i_n})$  are  $\mathcal{F}_{t'}^{i_n}$ -measurable  $(p^*(\mu_0, \mu_1) - \epsilon, \delta)$ -good estimators of  $S$ .  $\square$

## 4 Examples

In this section we give two examples showing that the assumptions of bounded out-degree and  $L$ -connectedness are crucial. To this end we need to construct equilibrium strategies where we can specify some of the responses. Our approach will be to describe the initial moves of the agents and then extend this to a (usual sense) equilibrium strategy profile.

Define the set of times and histories agents have to respond to as  $\mathcal{H} = \{(i, t, a) : i \in V, t \in \mathbb{N}_0, a \in [0, 1] \times \{0, 1\}^{|\partial i| \cdot t}\}$ . The set  $[0, 1] \times \{0, 1\}^{|\partial i| \cdot t}$  is interpreted as the pair of the private belief of  $i$  and the history of actions observed by agent  $i$  up to time  $t$ . If  $a \in [0, 1] \times \{0, 1\}^{|\partial i| \cdot t}$  then for  $0 \leq t' \leq t$  we let  $a_{t'} \in [0, 1] \times \{0, 1\}^{|\partial i| \cdot t'}$  denote the history restricted to times up to  $t'$ . We say a subset  $\mathcal{H} \in \mathcal{H}$  is *history-closed* if for every  $(i, t, a) \in \mathcal{H}$  we have that for all  $0 \leq t' \leq t$  that  $(i, t', a_{t'}) \in \mathcal{H}$ .

For a strategy profile  $\bar{Q}$  denote the optimal utility for  $i$  under any response as  $u_i^*(\bar{Q}) = \sup_{\bar{R}} u_i(\bar{R})$  where the supremum is over strategy profiles  $\bar{R}$  such that  $R^j = \bar{Q}^j$  for all  $j \neq i$  in  $V$ .

**Definition 4.1.** On a history-closed subset  $\mathcal{H} \in \mathcal{H}$  a forced response  $q_{\mathcal{H}}$  is a map  $q_{\mathcal{H}} : \mathcal{H} \rightarrow \{0, 1\}$  denoting a set of actions we force the agents to make. A strategy profile  $\bar{Q}$  is  $q_{\mathcal{H}}$ -forced if for every  $(i, t, a) \in \mathcal{H}$  if agent  $i$  at time  $t$  has seen history  $a$  from its neighbours then it selects action  $q_{\mathcal{H}}(i, t, a)$ . A strategy profile  $\bar{Q}$  is a  $q_{\mathcal{H}}$ -equilibrium if it is  $q_{\mathcal{H}}$ -forced and for every agent  $i \in V$  it holds that  $u_i(\bar{Q}) \geq u_i(\bar{R})$  for any  $q_{\mathcal{H}}$ -forced strategy profile  $\bar{R}$  such that  $R^j = \bar{Q}^j$  for all  $j \neq i$  in  $V$ .

The following lemma can be proved by a minor modification of Lemma 2.13 and so we omit the proof.

**Lemma 4.2.** *Let  $\mathcal{H} \in \mathcal{H}$  be history-closed and let  $q_{\mathcal{H}}$  be a forced response. There exists a  $q_{\mathcal{H}}$ -equilibrium.*

Having constructed  $q_{\mathcal{H}}$ -equilibria we then will want to show that they are equilibria. In order to do that we appeal to the following lemma.

**Lemma 4.3.** *Let  $\bar{Q}$  be a  $q_{\mathcal{H}}$ -equilibrium. Suppose that for every agent  $i$ , any strategy profile  $\bar{R}$  that attains  $u_i^*(\bar{Q})$  has that for all  $t$ ,*

$$\mathbb{P} \left[ \bar{Q}_t^i(I_i, A_{[0,t)}^{\partial i}) \neq \bar{R}_t^i(I_i, A_{[0,t)}^{\partial i}), (i, t, (I_i, A_{[0,t)}^{\partial i})) \in \mathcal{H} \right] = 0. \quad (9)$$

*Then  $\bar{Q}$  is an equilibrium.*

*Proof.* If  $\bar{Q}$  is not an equilibrium then by compactness there exists a strategy profile for  $\bar{R}$  that attains  $u_i^*$  and differs from  $\bar{Q}$  only for agent  $i$ . By equation (9) this implies that agent  $i$  following  $\bar{R}$  must take the same actions almost surely as if they were following  $\bar{Q}$  until the end of the forced moves. Hence it is  $q_{\mathcal{H}}$ -forced and so  $\bar{R}$  is a  $q_{\mathcal{H}}$ -equilibrium. It follows that  $i$  cannot increase the utility of  $\bar{Q}$ , which is therefore an equilibrium.  $\square$

In order to show that every agent follows the forced moves almost surely we now give a lemma which gives a sufficient condition for an agent to act myopically, according to its posterior distribution. For an equilibrium strategy profile  $\bar{Q}$  let  $\bar{Q}_{i,t,a}^\dagger$  be the strategy profile where the agents follow  $\bar{Q}$  except that if agent  $i$  has  $a = (I_i, A_{[0,t)}^{\partial i})$  then from time  $t$  onwards agent  $i$  acts myopically, taking action  $B_{t'}^i(G, \bar{Q}_{i,t,a}^\dagger)$  for time  $t' \geq t$ . We denote

$$Y_\ell = Y_\ell(i, t, a) := \mathbb{E} \left[ \mathbb{P} \left[ S = 1 \mid \mathcal{F}_{t+\ell}^i(G, \bar{Q}_{i,t,a}^\dagger) \right] - \frac{1}{2} \mid \mathcal{F}_t^i, a = (I_i, A_{[0,t)}^{\partial i}) \right].$$

We will show that the following are sufficient conditions for agent  $i$  to act myopically. For  $\ell \in \{1, 2, 3\}$  we set  $\mathcal{B}_\ell = \left\{ 2Y_0 > \frac{\lambda^2(\frac{1}{2} - Y_{\ell-1})}{1-\lambda} \right\}$  and we set

$$\mathcal{B}_4 = \left\{ 2Y_0 > \lambda^2(\frac{1}{2} - Y_2) + \frac{\lambda^3(\frac{1}{2} - Y_3)}{1-\lambda} \right\}.$$

Since  $\bar{Q}$  and  $\bar{Q}_{i,t,a}^\dagger$  are the same up to time  $t-1$  we have that  $\mathcal{F}_t^i(G, \bar{Q})$  is equal to  $\mathcal{F}_t^i(G, \bar{Q}_{i,t,a}^\dagger)$ . As  $Y_\ell$  is the expectation of a submartingale it is increasing. Hence, after rearranging we see that  $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_4$ .

**Lemma 4.4.** *Suppose that for strategy profile  $\bar{Q}$  agent  $i$  has an optimal response, such that for any  $\bar{R}$  such that  $R^j = Q^j$  for all  $j \neq i$  in  $V$  then  $u_i(\bar{Q}) \geq u_i(\bar{R})$ . Then for any  $t$ ,*

$$\mathbb{P} [A_t^i(G, \bar{Q}) \neq B_t^i, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4] = 0,$$

*that is, agent  $i$  acts myopically at time  $t$  under  $\bar{Q}$  almost surely, on the event  $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ .*

*Proof.* If agent  $i$  acts under  $\bar{Q}_{i,t,a}^\dagger$  then its utility from time  $t$  onwards given  $a$  is

$$\begin{aligned} u_{i,t,a}(\bar{Q}_{i,t,a}^\dagger) &:= (1-\lambda) \sum_{t'=t}^{\infty} \lambda^{t'} \mathbb{E} \left[ \mathbb{P} \left[ A_{t'}^i(G, \bar{Q}_{i,t}^\dagger) = S \right] \middle| \mathcal{F}_t^i, a = (I_i, A_{[0,t]}^{\partial i}) \right] \\ &\geq (1-\lambda) \lambda^t \left( \frac{1}{2} + Y_0 + \lambda \left( \frac{1}{2} + Y_1 \right) + \lambda^2 \left( \frac{1}{2} + Y_2 \right) + \frac{\lambda^3}{1-\lambda} \left( \frac{1}{2} + Y_3 \right) \right) \end{aligned}$$

under  $\bar{Q}_{i,t,a}^\dagger$ . Now assume that the action of agent  $i$  at time  $t$  under  $\bar{Q}$  is not the myopic choice. Then its utility is at most

$$\begin{aligned} u_{i,t,a}(\bar{Q}) &\leq (1-\lambda) \lambda^t \left( \frac{1}{2} - \left| \mathbb{P} \left[ S = 1 \middle| \mathcal{F}_t^i, a = (I_i, A_{[0,t]}^{\partial i}) \right] - \frac{1}{2} \right| \right. \\ &\quad \left. + \lambda \mathbb{E} \left[ \mathbb{P} \left[ A_{t+1}^i(G, \bar{Q}) = S \right] \middle| \mathcal{F}_t^i, a = (I_i, A_{[0,t]}^{\partial i}) \right] + \frac{\lambda^2}{1-\lambda} \right). \end{aligned}$$

We note that at time  $t+1$  the information available about  $S$  is the same under both strategies since the only difference is the choice of action by agent  $i$  at time  $t$ , hence as  $i$  takes the optimal action under  $\bar{Q}^\dagger$ ,

$$\frac{1}{2} + Y_1 = \mathbb{E} \left[ \mathbb{P} \left[ A_{t+1}^i(G, \bar{Q}_{i,t,a}^\dagger) = S \right] \middle| \mathcal{F}_t^i, a = (I_i, A_{[0,t]}^{\partial i}) \right] \geq \mathbb{E} \left[ \mathbb{P} \left[ A_{t+1}^i(G, \bar{Q}) = S \right] \middle| \mathcal{F}_t^i, a = (I_i, A_{[0,t]}^{\partial i}) \right].$$

Since  $\bar{Q}$  is optimal for  $i$  we have that

$$0 \geq u_{i,t,a}(\bar{Q}_{i,t,a}^\dagger) - u_{i,t}(\bar{Q}) \geq (1-\lambda) \lambda^t \left( 2Y_0 - \lambda^2 \left( \frac{1}{2} - Y_2 \right) - \frac{\lambda^3}{1-\lambda} \left( \frac{1}{2} - Y_3 \right) \right). \quad (10)$$

Condition (10) does not hold under  $\mathcal{B}_4$  so  $\mathbb{P} \left[ A_t^i(G, \bar{Q}) \neq B_t^i, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \right] = 0$ .  $\square$

## 4.1 The Royal Family

In the main theorem we require that the graph  $G$  not only be strongly connected, but also  $L$ -connected and have bounded out-degrees, which are local conditions. In the following example the graph is strongly connected, has bounded out-degrees, but is not  $L$ -connected. We show that for bounded private beliefs asymptotic learning does not occur in all equilibria<sup>2</sup>.

**Example 4.5.** Consider the the following graph. The vertex set is comprised of two groups of agents: a “royal family” clique of  $R$  agents who all observe each other, and  $n \in \mathbb{N} \cup \{\infty\}$  agents - the “public” - who are connected in an undirected chain, and in addition can all observe all the agents in the royal family. Finally, a single member of the royal family observes one of the public, so that the graph is strongly connected.

We choose that  $\mu_0$  and  $\mu_1$  so that  $\mathbb{P} \left[ Z_0^i \in (1, 2) \cup (-2, -1) \right] = 1$  and set the forced moves so that all agents act myopically at time 1. By Lemma 4.2 we can extend this to a forced equilibrium  $\bar{Q}$ . By Lemma 4.3 it is sufficient to show that no agent can achieve their optimum without choosing the myopic action in the first round. By our choice of  $\mu_0$  and  $\mu_1$  we have that

$$\left| \mathbb{P} \left[ S = 1 \middle| \mathcal{F}_0^i \right] - \frac{1}{2} \right| = \frac{e^{|Z_0^i|}}{1 + e^{|Z_0^i|}} - \frac{1}{2} \geq \frac{e}{1+e} - \frac{1}{2} \geq \frac{1}{5}.$$

<sup>2</sup>We draw on Bala and Goyal’s [4] *royal family* graph.



Hence in the notation of Lemma 4.4 we have that  $Y_0 \geq \frac{1}{5}$  when  $t = 0$  for all  $i$  and  $a$  almost surely. Moreover, after the first round all agents see the royal family and can combine their information. Since the signals are bounded it follows that for some  $c = c(\mu_0, \mu_1) > 0$ , independent of  $R$  and  $n$

$$\mathbb{E} \left[ \frac{1}{2} - \left| \mathbb{P}[S = 1 | \mathcal{F}_1^i] - \frac{1}{2} \right| \middle| \mathcal{F}_0^i \right] \leq e^{-cR}.$$

Hence if  $R$  is a large constant  $\mathcal{B}_2$  holds so by Lemma 4.4 if an agent is to attain its maximal utility given the actions of the other agents it must act myopically almost surely at time 0. Thus  $\bar{Q}$  is an equilibrium.

Let  $\mathcal{J}$  denote the event that all agents in the royal family have a signal favouring state 1. On this event under  $\bar{Q}$  all agents in the royal family choose action 1 at time 0 and this is observed by all the agents so  $\mathcal{J} \in \mathcal{F}_1^i$  for all  $i$ . Since agents observe at most one other agent this signal overwhelms their other information and so

$$\mathbb{P}[S = 1 | \mathcal{F}_1^i, \mathcal{J}] \geq 1 - e^{-cR},$$

for all  $i \in V$ . Thus if  $R$  is a large constant  $\mathcal{B}_1$  holds for all the agents at time 1 so by Lemma 4.4 they all act myopically and choose action 1 at time 1. Since  $\mathcal{J} \in \mathcal{F}_1^i$  they also all knew this was what would happen so gain no extra information. Iterating this argument we see that all agents choose 1 in all subsequent rounds. However,  $\mathbb{P}[\mathcal{J}, S = 0] \geq e^{-c'R}$  where  $c'$  is independent of  $R$  and  $n$ . Hence as we let  $n$  tend to infinity the probability of learning does not tend to 1, and when  $n$  equals infinity the probability of learning does not equal 1.

## 4.2 The Mad King

More surprising is that there exist *undirected* (i.e., 1-connected) graphs with equilibria where asymptotic learning fails; These graphs have unbounded out-degrees. Note that in the myopic case learning is achieved on these graphs [19], and so this is an example in which strategic behavior impedes learning.

In this example we consider a finite graph which includes 5 classes of agents. There is a king, labeled  $u$ , and a regent labeled  $v$ . The court consists of  $R_C$  agents and the bureaucracy of  $R_B$  agents. The remaining  $n$  are the people. Note again that the graph is undirected.

- The king is connected to the regent, the court and the people.
- The regent is connected to the king and to the bureaucracy.
- The members of the court are each connected only to the king.
- The members of the people are each connected only to the king.
- The members of the bureaucracy are each connected only to the regent.

See Figure 1.

As in the previous example we will describe some initial forced equilibrium and then appeal to existence results to extend it to an equilibrium. We suppose that  $\mu_0$  and  $\mu_1$  are such that  $\mathbb{P}[Z_0^i \in (1, 1 + \epsilon) \cup (-\sqrt{7}, -\sqrt{7} + \epsilon)] = 1$  where  $\epsilon$  is some very small positive constant, and will choose  $R_C, \lambda$  and  $R_B$  so that  $e^{R_C}$  is much smaller than  $\frac{1}{1-\lambda}$  which in turn will be much smaller than  $R_B$ :

$$e^{R_C} \ll \frac{1}{1-\lambda} \ll R_B.$$

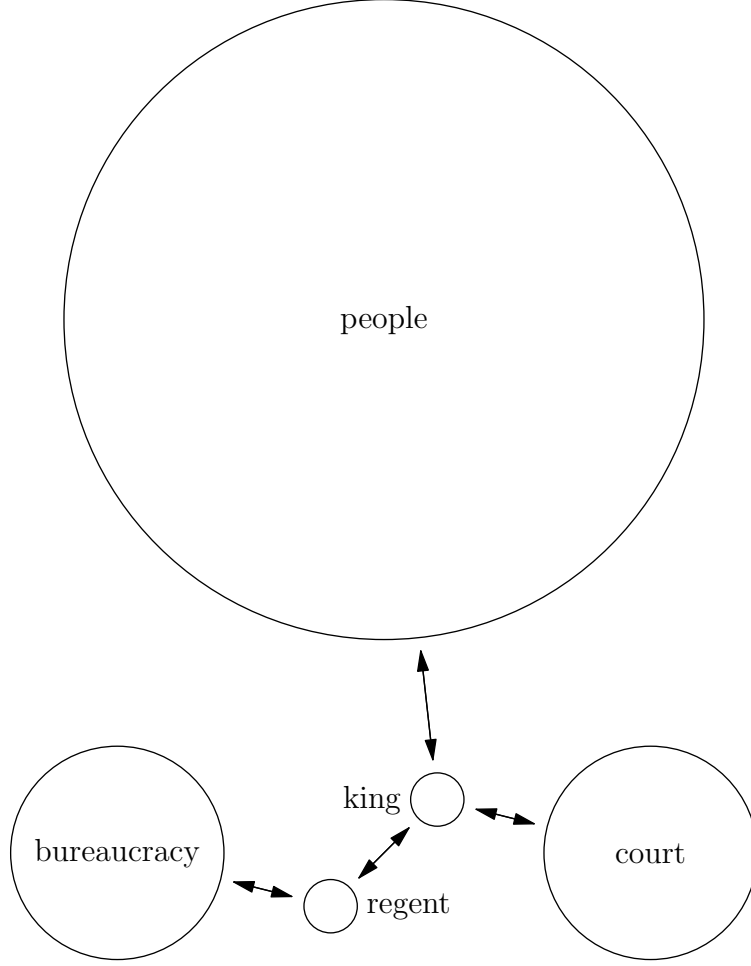


Figure 1: The Mad King.

The equilibrium we describe will involve the people being forced to choose action 0 in rounds 0 and 1, as otherwise the king “punishes” them by withholding his information. As an incentive to comply he offers them the opinion of his court and, later, of his bureaucracy. While the opinion of the bureaucracy is correct with high probability, is it still bounded, and so, even as the size of the public tends to infinity, the probability of learning stays bounded away from one.

We now describe a series of forced moves for the agents, fixing  $\delta > 0$  to be some small constant.

- The regent acts myopically at time 0. If for some state  $s$   $\mathbb{P}[S = s | \mathcal{F}_1^v] \geq 1 - e^{-\delta R_B}$  then the regent chooses states  $s$  in round 1 and all future rounds, otherwise his moves are not forced.
- The king acts myopically in rounds 0 and 1 unless one or more of the people chooses action 1 in round 0 or 1 in which case, enraged, he chooses action 1 in all future rounds. Otherwise if  $s$  is the action of the regent at time 1 then from time 2 the king takes action  $s$  until the regent deviates and chooses another action.
- The members of the bureaucracy act myopically in round 0 and 1. If  $s$  is the action of the regent at time 1 then from time 2 the members of the bureaucracy take action  $s$  until the regent deviates and chooses another action.

- The members of the court act myopically in round 0 and 1. At time 2 they copy the action of the king from time 1. If  $s$  is the action of the king at time 2 then from time 3 the members of the bureaucracy take action  $s$  until the king deviates and chooses another action.
- The people choose action 0 in rounds 1 and 2. At time 2 they copy the action of the king from time 1. If  $s$  is the action of the king at time 2 then from time 3 the people take action  $s$  until the king deviates and chooses another action.

By Lemma 4.2 this can be extended to a forced equilibrium strategy  $\bar{Q}$ . We will show that this is also an equilibrium strategy in the unrestricted game by establishing equation (9). In what follows when we say acts optimally or in an optimal strategy we mean for an agent with respect to the actions of the other agents under  $\bar{Q}$ .

First consider the regent. By our choice of  $\mu_0, \mu_1$  we have that  $Y_0 > \frac{1}{5}$ . Let  $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$  where  $\mathcal{J}_s$  denotes the event that  $\mathbb{P}[S = s | \mathcal{F}_1^v] \geq 1 - e^{-\delta R_B}$ . Since the regent views all the myopic actions of the bureaucracy it knows the correct value of  $S$  except with probability exponentially small in  $R_B$  so for  $s \in \{0, 1\}$ , if  $\delta > 0$  is small enough,

$$\mathbb{P}[\mathcal{J}_s | S = s] \geq 1 - e^{-\delta R_B}$$

and hence for large enough  $R_B$  we have that  $Y_1 \geq \frac{1}{2} - 2e^{-\delta R_B}$  which implies that  $\mathcal{B}_2$  holds at time 1. By Lemma 4.4 in any optimal strategy the regent acts myopically in round 0, and so follows the forced move. On the event  $\mathcal{J}_s$  the regent follows  $s$  in all future steps. At time 1 condition  $\mathcal{B}_1$  holds so again the regent follows the forced move in any optimal strategy. We next claim that for large enough  $R_B$

$$\mathbb{P}\left[\mathbb{P}[S = s | \mathcal{F}_2^v] \geq 1 - e^{-\delta R_B/2} \middle| \mathcal{J}_s\right] = 1 \quad (11)$$

Assuming (11) holds then condition  $\mathcal{B}_1$  again holds so the regent must choose  $s$  at time 2 in any optimal strategy. By construction of the forced moves from time 2 onwards the king and bureaucracy simply imitate the regent and so it receives no further information from time 2 onwards. Thus again using Lemma 4.4 we see that under any optimal strategy the regent must follow its forced moves.

To establish that the regent follows the forced moves in any optimal strategy it remains to show that Condition (11) holds. The information available to the regent at time 2 includes the actions of the king and the bureaucracy at times 0 and 1. Consider the actions of the bureaucracy at times 0 and 1. At time 0 it follows its initial signal. At time 1 they also learn the initial action of the regent who acts myopically. By our assumption on  $\mu_0$  and  $\mu_1$  that  $\mathbb{P}[Z_0^i \in (1, 1 + \epsilon) \cup (-\sqrt{7}, -\sqrt{7} + \epsilon)] = 1$ , an initial signal towards 0 is much stronger than an initial signal towards 1, since whenever  $Z$  is negative it is at most  $-\sqrt{7} + \epsilon$ . For  $i$ , a member of the bureaucracy, we have that  $Z_1^i \geq 2$  if both  $i$  and the regent choose action 1 at time 1. However, if either  $i$  or the regent choose action 0 at time 1 then  $Z_t^i \leq -\sqrt{7} + \epsilon + 1 + \epsilon < -1$ . Since the actions of  $i$  and the regent at time 0 are known to the regent at time 1, he gains no extra information at time 2 from his observation of  $i$  at time 1 since he can correctly predict its action.

The information the regent has available at time 2 is thus his information from time 1 together with the information from observing the king. The information available to the king is a function of his initial signal and that of the regent and the court. Since this is only  $R_C + 1$  members and we choose  $R_B$  to be much larger than  $R_C$  it is insignificant compared to the information the regent observed from the court at time 0 and hence (11) holds. Thus, there is no optimal strategy for the regent that deviates from the forced moves.

As we noted above the members of the bureaucracy have  $|Z_0^i|, |Z_1^i| \geq 1$  almost surely. For  $t \geq 1$  let  $\mathcal{M}_{s,t}$  denote the event that the regent chose action  $s$  for times 1 up to  $t$ . As argued above,

$\mathcal{J}_s \subset \mathcal{M}_{s,t}$  for all  $t$  under  $\bar{Q}$ . This analysis holds even if a single member of the bureaucracy adopts a different strategy as we have taken  $R_B$  to be large so this change is insignificant. Given that  $\mathcal{M}_{s,t}$  holds, the only additional information available to agent  $i$ , a member of the bureaucracy, is their original signal and the action at time 1 of the regent. Thus

$$\mathbb{P}[S = s | \mathcal{F}_t^i, \mathcal{M}_{s,t}] \geq 1 - e^{-\delta R_B/2}.$$

It follows then by Lemma 4.4 that acting myopically at times 0 and 1 and then imitating the regent until he changes his action is the sole optimal strategy for a member of the bureaucracy.

Next consider the forced responses of the king. Since under  $\bar{Q}$  the people always choose action 0 at times 0 and 1, the rule forcing the king to choose action 1 after seeing a 1 from the people is never invoked. We claim that, provided  $R_B$  is taken to be sufficiently large, that the king acts myopically at times 0 and 1. At time 0 the posterior probability of  $S = 1$  is bounded away from  $1/2$  so  $Y_0$  is bounded away from 0 while  $\frac{1}{2} - Y_2 \leq 2e^{-\delta R_B/2}$  so by Lemma 4.4 the king must act myopically. Similarly at time 1 since our choice of  $\mu_0$  and  $\mu_1$  to have their log-likelihood ratio concentrated around either 1 or  $-\sqrt{7}$  a posterior calculation gives that,

$$|Z_1^u - \#\{i \in \partial u : A_0^i(\bar{Q}) = 1\} + \sqrt{7}\#\{i \in \partial u : A_0^i = 0\}| \leq \epsilon(2 + R_C)$$

and thus for some  $\epsilon(R_C) > 0$  sufficiently small we can find an  $\epsilon'(\epsilon, R_C) > 0$  such that  $Y_0 = \left| \frac{e^{Z_1^u}}{1 + e^{Z_1^u}} - \frac{1}{2} \right| > \epsilon'$ . Since we again have that  $\frac{1}{2} - Y_1 \leq 2e^{-\delta R_B/2}$  taking  $R_B = R_B(\epsilon, R_C)$  to be sufficiently large  $\mathcal{B}_2$  holds and so the king must act myopically. It remains to see that the king should imitate the the regent from time 2 onwards unless the regent subsequently changes his action in any optimal strategy. This follows from a similar analysis to the case of the members of the bureaucracy so we omit it.

We next move to an agent  $i$ , a member of the court. At time 0 the agent has  $Y_0 > \frac{e}{1+e} - \frac{1}{2} > \frac{1}{5}$ . Agent  $i$  at time 1 views the action of the king who has in turn viewed the actions of the whole court at time 0 so  $\frac{1}{2} - Y_2 \leq e^{-cR_C}$ . At time 2 the agent sees the action of the king who has imitated the action of the regent at time 1 so  $\frac{1}{2} - Y_3 \leq e^{-\delta R_B/2}$ . Hence provided that  $R_C$  is sufficiently large and  $R_B(R_C, \lambda)$  is sufficiently large then  $\mathcal{B}_4$  holds and  $i$  must act myopically at time 0. The information of a member of the court at time 1 is a combination of their initial signal and the action of the king at time 1. Similarly to a member of the bureaucracy, by the choice of  $\mu_0$  and  $\mu_1$  we have that  $|Z_1^i| \geq 1$  and so  $Y_0 > \frac{1}{5}$ . Also  $\frac{1}{2} - Y_2 \leq e^{-\delta R_B/2}$  since this includes the information from the action of the regent at time 1. Thus  $\mathcal{B}_3$  holds and  $i$  must act myopically at time 1. At time 2 agents  $i$  knows the action of the king from round 2 so  $Y_0 \geq \frac{1}{2} - e^{-cR_C}$  and  $\frac{1}{2} - Y_1 \leq e^{-\delta R_B/2}$  so  $\mathcal{B}_2$  holds and  $i$  must act myopically at time 2. Finally from time 3 onwards agent  $i$  knows the action of the regent at time 1. As with the king and bureaucracy this will not be changed unless  $i$  receives new information, that is the king changes his action sometime after time 2. Thus any optimal strategy of  $i$  follows the forced moves.

This finally leaves the people. Let agent  $i$  be one of the people. We first check that it is always better for them to wait and just say 0 in rounds 0 and 1 in order to get more information from the king, their only source. If agent  $i$  chooses action 1 at time 0 then the total information it receives is a function of the initial signals of  $i$  and the king. Thus, since the signals are uniformly bounded, even if the agent knew the signals exactly we would have that for some  $c'(\mu_0, \mu_1)$  that the utility from such a strategy is at most  $1 - e^{-2c'}$ . If the agents acts with 0 at time 0 but 1 at time 1 it can potentially receive information from the initial signals of the king, court and regent as well as it's own but still the optimal utility even using all of this information is at most  $1 - e^{-c'(R_C+3)}$ . Consider instead the expected utility following the forced moves. On the event  $\mathcal{J}$  agent  $i$  will have utility at least  $\lambda^3(1 - e^{-\delta R_B})$  which is greater than  $1 - e^{-c'(R_C+3)}$  provided that  $\lambda$  is sufficiently

close to 1 and  $R_B$  is sufficiently large. Thus agent  $i$  must choose action 0 at times 0 and 1 in any optimal strategy. The analysis of rounds 2 and onwards follows similarly to the court and thus any optimal strategy of  $i$  follows all the forced moves.

This exhaustively shows that there is no alternative optimal strategy for any of the agents which differs from the forced moves. Thus  $\bar{Q}$  is an equilibrium. However, on the event  $\mathcal{J}_1$  all the agents actions converge to 1. However,  $\mathbb{P}[\mathcal{J}, S = 0] \geq e^{-c'' R_B} > 0$  where  $c''$  is independent of  $R_C, R_B, \lambda$  and  $n$ . Hence, as we let  $n$  tend to infinity the probability of learning does not tend to 1.

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